Final Exam

Monday 12/15 from 3:30-5:30 PM
Elliott Hall (Exact Seating TBD)

Format: 2 hrs, 24 questions

12 Questions	Other 12 Questions
From Exams 1-3 More specifically • 1 question comes <u>directly</u> from Exams 1, 2, and 3 [3 questions total] • 3 questions comes <u>directly</u> from the Exam Problem Sets on Achieve [9 Total]	Covers the content after Exam 3 (Lectures 29-35)

Problems for Day 1 (Lectures 1-15): Limits/Continuity, Derivatives up to Related Relates

1. Compute the following limits:

   (a) $\lim_{x\to 1} \sqrt{5x+6}$

   Continuous

$$\lim_{x\to 1} \sqrt{5x+6} = \sqrt{5(1)+6} = \sqrt{11}$$

   (b) $\lim_{x\to 4} \frac{x^3-7x^2+12x}{4-x}$

$$\lim_{x\to 4} \frac{x^3-7x^2+12x}{4-x} = \lim_{x\to 4} \frac{x(x-3)\cancel{(x-4)}}{-\cancel{(x-4)}}$$

$$= \lim_{x\to 4} \frac{x(x-3)}{-1} = \frac{4(4-3)}{-1} = -4$$

$f(x) = \sqrt{5x}$

   (c) $\lim_{h\to 0} \frac{\sqrt{5(x+h)}-\sqrt{5x}}{h}$, $x > 0$

$$\lim_{h\to 0} \frac{\sqrt{5(x+h)}-\sqrt{5x}}{h} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \overset{\text{def}}{=} f'(x)$$

Ie, we need to find $[\sqrt{5x}]'$

Power Rule

$[x^n]' = nx^{n-1}$

Chain Rule:

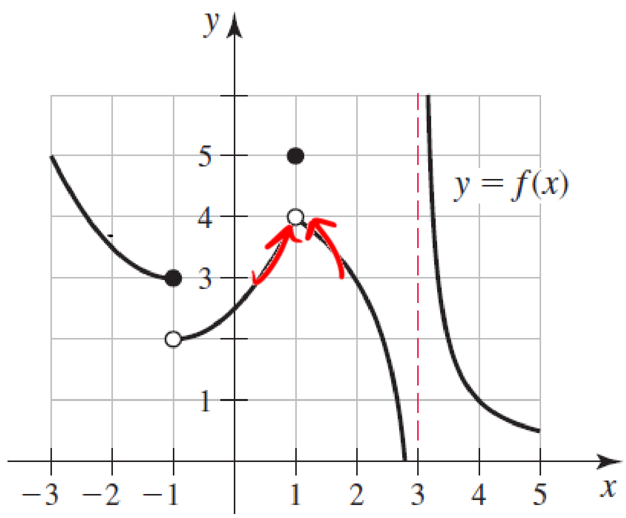
$[f(g(x))]' = f'(g(x)) \cdot g'(x)$

$$\left[(5x)^{\frac{1}{2}}\right]' = \frac{1}{2}(5x)^{\frac{1}{2}-1} \cdot (5x)'$$

$$= \frac{1}{2}(5x)^{-\frac{1}{2}} \cdot 5$$

$$= \frac{5}{2\sqrt{5x}}$$

2. The graph of a function f is given below. Locate and classify all discontinuities of f.



Disc @ $x = -1, 1, 3$

Near $x = -1$:

$\lim_{x \to -1^-} f(x) = 3$ | Fcn defined but

$\lim_{x \to -1^+} f(x) = 2$ | limit DNE $\Rightarrow$ Jump Discontinuity

Near $x = 1$: $\lim_{x \to 1} f(x) = 4$

$f(1) = 5$ $\Rightarrow$ Removable Discontinuity

Near $x = 3$:

$\lim_{x \to 3^+} f(x) = \infty \Rightarrow$ V. asymptote

$\lim_{x \to a} f(x) = f(a)$

3. Find y' if:

(a) $y = x^2 + 2x + 9$

$$y' = [x^2]' + [2x]' + [9]'$$
$$= 2x^{2-1} + 2 \cdot 1x^{1-1} + 0$$
$$= 2x + 2$$

(b) $y = 5t^2 \sin t$

$$[f(x)\,g(x)]' = f'(x)\,g(x) + f(x)\,g'(x)$$

$$y = 5t^2 (\sin t)$$
$$y' = 10t (\sin t) + 5t^2 \cos t$$

(c) $y = \frac{e^x}{1+\cos x}$

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

$$y = \frac{e^x}{1+\cos x} \quad \begin{matrix} \leftarrow f \\ \leftarrow g \end{matrix}$$

$$y' = \frac{e^x (1+\cos x) - e^x(-\sin x)}{(1+\cos x)^2}$$
$$= \frac{e^x (1+\cos x + \sin x)}{(1+\cos x)^2}$$

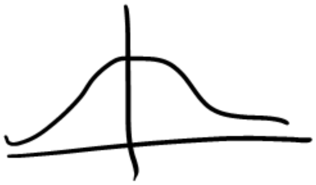
(d) $y = (x+1)^{\frac{1}{x}}$

$$[\ln x]' = \frac{1}{x}$$
$$[\ln f(x)]' = \frac{1}{f(x)} \cdot f'(x)$$

$$\ln(y) = \ln\left((x+1)^{\frac{1}{x}}\right)$$
$$\ln(y) = \frac{\ln(x+1)}{x}$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{\frac{1}{x+1}[x+1]' \cdot x - \ln(x+1)}{x^2}$$

$$\frac{dy}{dx} = y\left[\frac{\frac{x}{x+1} - \ln(x+1)}{x^2}\right]$$
$$\frac{dy}{dx} = (x+1)^{\frac{1}{x}}\left[\frac{\frac{x}{x+1} - \ln(x+1)}{x^2}\right]$$

4. Given the algebraic curve $x^2y + y^3 = 5$, find the slope of the tangent line at $(2, 1)$.



$$x^2 y + y^3 = 5$$

$$\frac{d}{dx}\left(\underbrace{x^2}_{f}\underbrace{[y_{(x)}]}_{g} + \underbrace{[\underbrace{y_{(x)}}_{g}]^3}_{f}\right) = \frac{d}{dx}(5)$$

$$2xy + x^2\frac{dy}{dx} + 3y^2 \cdot \frac{dy}{dx} = 0$$

$$x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = -2xy$$

$$\frac{dy}{dx}(x^2 + 3y^2) = -2xy$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 + 3y^2}$$

$$\left.\frac{dy}{dx}\right]_{\substack{x=2\\y=1}} = \frac{-2(2)(1)}{(2)^2 + 3(1)^2} = -\frac{4}{7}$$

$\frac{\ln t}{t^2}$

5. The position (in meters) of a particle traveling in a straight time is given by $s(t) = \frac{\ln x}{x^2}$, where t is the time in seconds. Find the acceleration after 1 second.

$$s(t) = \frac{\ln t}{t^2}$$

$\ln t \to f$

$t^2 \to g$

$$v(t) = \frac{ds}{dt} = \frac{\frac{1}{t} \cdot t^2 - 2t \ln(t)}{t^4} = \frac{t - 2t \ln t}{t^4}$$

$$= \frac{t(1 - 2 \ln t)}{t^4} = \frac{1 - 2 \ln t}{t^3}$$

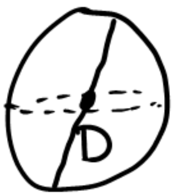
$$a(t) = \frac{dv}{dt} = \frac{\left(0 - 2\left(\frac{1}{t}\right)\right)t^3 - 3t^2(1 - 2 \cdot \ln t)}{t^6}$$

$$= \frac{-2t^2 - 3t^2 + 6t^2 \ln t}{t^6} = \frac{-5t^2 + 6t^2 \ln t}{t^6}$$

$$= \frac{-t^2(5 - 6 \ln t)}{t^6} = -\frac{5 - 6 \ln t}{t^4}$$

$$a(1) = -\frac{5 - 6(0)}{1^4} = -\frac{5 - 0}{1} = -5 \frac{m}{s^2}$$

6. A spherical balloon is inflated at a rate of 10 cm^3/min. At what rate is the **diameter** of the balloon increasing when the balloon has a diameter of 5 cm?



Need to know: $\frac{dD}{dt}$ when $D=5$

Know: If V is volume, $\frac{dV}{dt}=10$

Formula: $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 = \frac{4}{3}\pi \cdot \frac{D^3}{8}$

$$V = \frac{\pi}{6}\left[D(t)\right]^3$$

$$\frac{dV}{dt} = \frac{\pi}{6} \cdot 3D^2 \cdot \frac{dD}{dt}$$

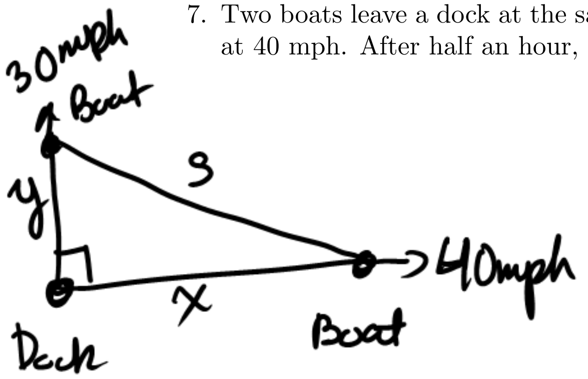
$$10 = \frac{\pi}{6} \cdot 3[5]^2 \cdot \frac{dD}{dt}$$

$$10 = 25\left(\frac{\pi}{2}\right)\frac{dD}{dt}$$

$$20 = 25\pi \frac{dD}{dt}$$

$$\frac{dD}{dt} = \frac{20}{25\pi} \approx 0.25 \ \frac{\text{cm}}{\text{min}}$$

7. Two boats leave a dock at the same time. One boat travels north at 30 mph and the other travels east at 40 mph. After half an hour, how fast is the distance between the boats increasing?



Need to know: $\frac{ds}{dt}$ when $t = \frac{1}{2}$

Know: $\frac{dx}{dt} = 40$; $\frac{dy}{dt} = 30$

Formula: $x^2 + y^2 = s^2$

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(s^2)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2s\frac{ds}{dt}$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = s\frac{ds}{dt}$$

Find x, y, and s at $t = \frac{1}{2}$

$$x = 40t \Rightarrow x(\tfrac{1}{2}) = 40(\tfrac{1}{2}) = 20$$

$$y = 30t \Rightarrow y(\tfrac{1}{2}) = 30(\tfrac{1}{2}) = 15$$

$$s = \sqrt{x^2 + y^2}$$

$$s(\tfrac{1}{2}) = \sqrt{20^2 + 15^2} = \sqrt{5^2 \cdot 4^2 + 5^2 \cdot 3^2} = \sqrt{5^2(16+9)}$$

$$= \sqrt{5^2(25)} = 25$$

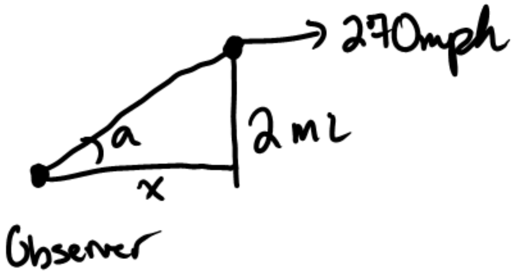
$$x\frac{dx}{dt} + y\frac{dy}{dt} = s\frac{ds}{dt}$$

$$(20)(40) + 15(30) = 25\frac{ds}{dt}$$

$$1250 = 25\frac{ds}{dt}$$

$$\frac{ds}{dt} = 50 \text{ mph}$$

8. A plane is flying away from you at a speed of 270 mph at a constant altitude of 2 miles. Find the rate at which the angle of elevation is decreasing when the angle is $\pi/6$.



Need to know: $\frac{da}{dt}$ when $a = \frac{\pi}{6}$

Know: $\frac{dx}{dt} = 270$

Formula:

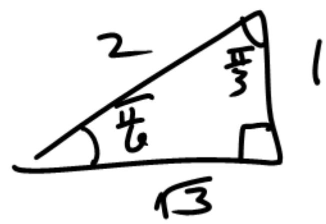
$$\frac{x}{2} = \cot a$$

$$x = 2 \cot a$$

$$\frac{dx}{dt} = 2\left[-\csc^2 a\right] \cdot \frac{da}{dt}$$

$$\sin^2 a \frac{dx}{dt} = -2 \frac{da}{dt}$$

$$\frac{da}{dt} = -\frac{1}{2} \sin^2 a \left(\frac{dx}{dt}\right)$$



$$\frac{da}{dt} = -\frac{1}{2}(270)\left(\frac{1}{2}\right)^2$$

$$= -\frac{270}{8} = -\frac{135}{4} \frac{\text{rad}}{\text{hr}}$$

It is decreasing at a rate of $\frac{135}{4} \frac{\text{rad}}{\text{hr}}$