Problems for Day 2 (Lectures 16-28): 1st/2nd Derivative Test, Optimization, Riemann Sums

1. Let $f(x) = \frac{x^2+3}{x-1}$

   $\frac{x^2+3}{x-1}$

   (a) Locate all (if any) asymptotes.

VAs: $x-1 \overset{\text{set}}{=} 0$

$\boxed{x=1}$

HAs: Occur when deg top $\leq$ deg bottom

$2 \not\leq 1$

Slant:

$\boxed{y = x+1}$

$$\begin{array}{r} x+1 \\ x-1 \overline{) x^2+3} \\ -x^2-x \\ \hline x+3 \\ -x-1 \\ \hline 4 \end{array}$$

(b) Find all critical values.

$$f(x) = \frac{x^2+3}{x-1}$$

$[x-1]'$

$$f'(x) = \frac{2x(x-1) - (x^2+3)(1)}{(x-1)^2} = \frac{x^2-2x+3}{(x-1)^2}$$

$$= \frac{(x+1)(x-3)}{(x-1)^2} \overset{\text{set}}{=} 0$$

$$(x+1)(x-3) = 0$$

$$x = -1, 3$$

$\frac{1}{x^2}$

$$\frac{(x+1)(x-3)}{(x-1)^2}$$

(c) When is f increasing? When is f decreasing?

$$\longleftarrow \; -1 \quad 1 \quad 3 \; \longrightarrow$$

Test Point	-100	0	2	100
Sign of $(x-1)^2$	$+$	$+$	$+$	$+$
Sign of $x+1$	$-$	$+$	$+$	$+$
Sign of $x-3$	$-$	$-$	$-$	$+$
Sign of f'	$+$	$-$	$-$	$+$
Results of I/D Test	Inc ↗	↘ Dec ↘	↘ Dec ↘	↗ Inc

Inc : $(-\infty,-1)\cup(3,\infty)$

Dec $(-1,1)\cup(1,3)$

(d) Use the 1st Derivative Test to determine the locations of any local maximums/minimums.

Local Max

Local Min

Local Max at $x=-1$

Local Min at $x=3$

(e) Verify your answer using the 2nd derivative test.

$$f'(x) = \frac{x^2 - 2x + 3}{(x-1)^2}$$

$$[x-1]'$$

$$f''(x) = \frac{(2x-2)(x-1)^2 - (x^2-2x+3)\,2(x-1)(1)}{(x-1)^4}$$

$$f''(x) = \frac{8(x-1)}{(x-1)^4} = \frac{8}{(x-1)^3}$$

$$f''(-1) = \frac{8}{(-1-1)^3} = -1 < 0 \Rightarrow \text{Local Max at } x = -1$$

$$f''(3) = \frac{8}{(2-1)^3} = 1 > 0 \Rightarrow \text{Local Min at } x = 3$$

(f) Find all inflection points (if any).

$$f''(x) = \frac{8}{(x-1)^3} \overset{\text{set}}{=} 0$$

$$\Rightarrow \quad 8 = 0 \quad \text{No Solution!}$$

Still could change concavity at asymptote

(g) When is f concave up? When is f concave down?

$\frac{8}{(x-1)^3}$

1

Test Point	0	100
Sign of f''	$-$	$+$
Results of C Test	CD	CU

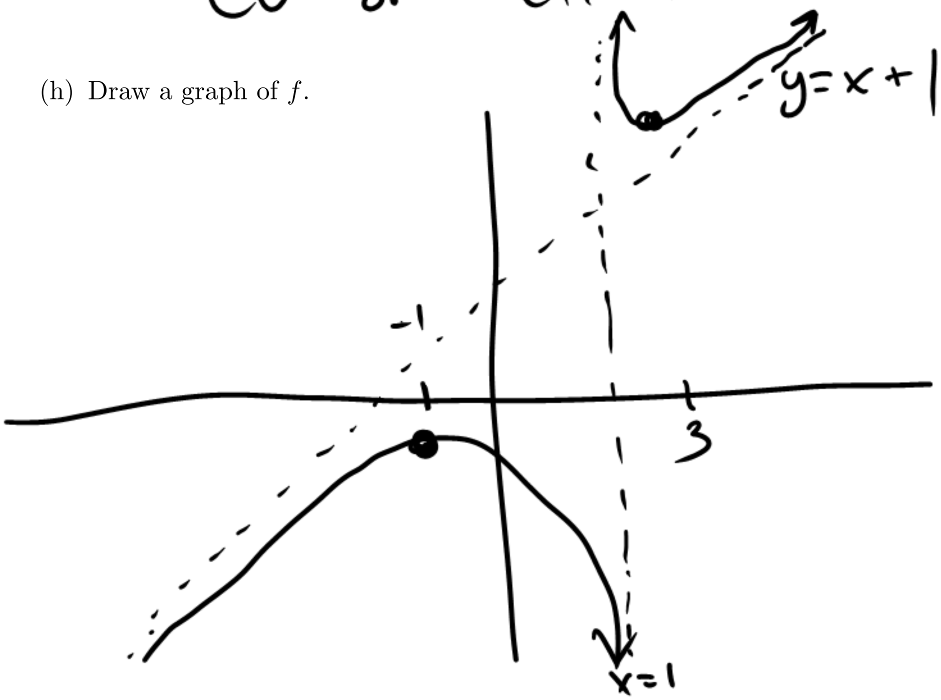
$++$

$--$

CD on $(-\infty, 1)$

CU on $(1, \infty)$

(h) Draw a graph of f.

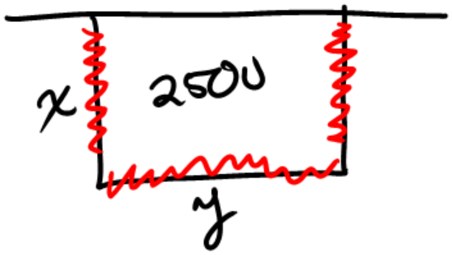


2. Compute $\lim\limits_{x\to\infty} \frac{2x^2-4}{5x^2-4x+6}$

$$\lim_{x\to\infty} \frac{2x^2-4}{5x^2-4x+6} = \lim_{x\to\infty} \frac{x^2\left(2-\frac{4}{x^2}\right)}{x^2\left(5-\frac{4}{x}+\frac{6}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{\overset{\to 2}{2}-\overset{\to 0}{\frac{4}{x^2}}}{\underset{\to 5}{5}-\underset{\to 0}{\frac{4}{x}}+\underset{\to 0}{\frac{6}{x^2}}} = \frac{2-0}{5-0+0} = \boxed{\frac{2}{5}}$$

3. (Exam 3, Problem 8) A family wants to fence a rectangular play area alongside the wall of their house. The wall of their house bounds one side of the play area. If they want the play area to be exactly 2500 ft^2, what is the least amount of fencing needed? Round your answer to the nearest tenth place.



Obj Minimize $P(x,y) = 2x + y$

Given $xy = 2500$

$y = \frac{2500}{x}$

$$P(x) = 2x + \frac{2500}{x}$$

$$P'(x) = 2 - \frac{2500}{x^2} \underset{\text{Set}}{=} 0$$

$$2 = \frac{2500}{x^2}$$

$$\frac{x^2}{2500} = \frac{1}{2}$$

$$x^2 = 1250 \;;\; x > 0$$

Potential Loc for a min

$$x = \sqrt{1250} = \sqrt{5^4 \cdot 2} = 25\sqrt{2}$$

Verify it's a min

$$y = \frac{2500}{x} \Rightarrow y = \frac{2500}{25\sqrt{2}} = \frac{100}{\sqrt{2}} = 50\sqrt{2}$$

$$P(25\sqrt{2}, 50\sqrt{2}) = 2(25\sqrt{2}) + 50\sqrt{2} = 100\sqrt{2} \approx 141.4 \text{ ft}$$

4. (Exam 3, Problem 9) A company's marketing department has determined that if their product is sold at the price of p dollars per unit, they can sell $q = 1200 - 100p$ units. Each unit costs 6 dollars to make. What price, p, should the company charge to maximize their profit?

$$\text{Profit} = \text{Revenue} - \text{Costs} = \left(\frac{\text{Price}}{\text{Item}}\right)\left(\begin{matrix}\#\text{ items}\\ \text{sold}\end{matrix}\right) - \left(\frac{\text{Cost}}{\text{item}}\right)\left(\begin{matrix}\#\text{ of}\\ \text{items sold}\end{matrix}\right)$$

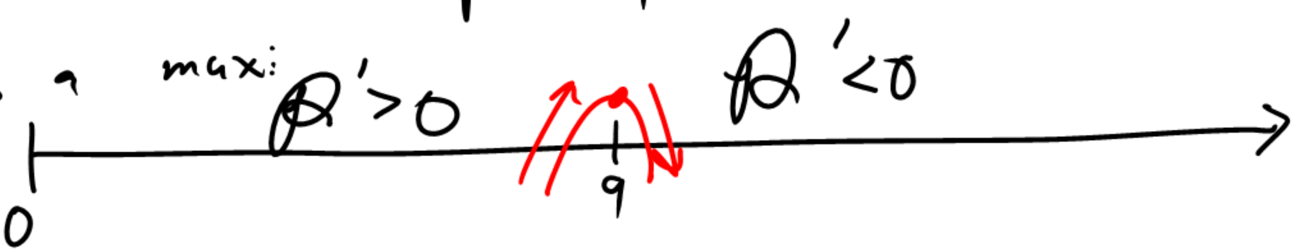
$$P_{(p,q)} = pq - 6q$$

Obj: Max $P_{(p,q)} = pq - 6q$

Given $q = 1200 - 100p$

$$P_{(p)} = p(1200 - 100p) - 6(1200 - 100p)$$
$$= 1200p - 100p^2 - 7200 + 600p$$
$$P_{(p)} = -100p^2 + 1800p - 7200$$
$$\frac{dP}{dp} = -200p + 1800 \overset{set}{=} 0$$
$$-200p = -1800$$
$$p = 9$$

Verify it's a max:



The company should sell their product at \$9.

5. Solve the IVP:

$$\begin{cases} f'(t) = \sin t + 2t \\ f(0) = 5 \end{cases}$$

$$\frac{2t^{1+1}}{1+1}$$

$$f'(t) = \sin t + 2t$$

$$\int f'(t)\,dt = \int (\sin t + 2t)\,dt$$

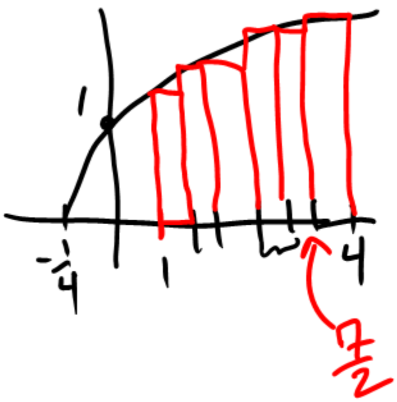
$$f(t) = -\cos t + t^2 + C$$

$\underline{f(0)=5:}$

$$f(0) = -1 + 0 + C \overset{\text{set}}{=} 5$$

$$C = 6$$

6. Write the left and right Riemann Sums to estimate $\int_1^4 \sqrt{4x+1}\, dx$ when $N = 6$.



$$\Delta x = \frac{b-a}{N} = \frac{4-1}{6} = \frac{1}{2}$$

$$L_N = \sum_{i=0}^{N-1} f(\underset{1}{a} + i\underset{\frac{1}{2}}{\Delta x})\Delta x$$

$$L_6 = \sum_{i=0}^{5} \sqrt{4\left(1+i\left(\tfrac{1}{2}\right)\right)+1}\left(\frac{1}{2}\right) = \frac{1}{2}\sum_{i=0}^{5}\sqrt{4\left(1+\tfrac{i}{2}\right)+1}$$

$$= \frac{1}{2}\left[\underbrace{\sqrt{4(1)+1}}_{i=0} + \underbrace{\sqrt{4(\tfrac{3}{2})+1}}_{i=1} + \underbrace{\sqrt{4(2)+1}}_{i=2} + \underbrace{\sqrt{4(\tfrac{5}{2})+1}}_{i=3} + \underbrace{\sqrt{4(3)+1}}_{i=4} + \underbrace{\sqrt{4(\tfrac{7}{2})+1}}_{i=5}\right]$$

$$\approx 9.3385$$

$$R_6 = \frac{1}{2}\sum_{i=1}^{6}\sqrt{4\left(1+\tfrac{i}{2}\right)+1}$$

$$= \frac{1}{2}\left[\underbrace{\sqrt{4(\tfrac{3}{2})+1}}_{i=1} + \underbrace{\sqrt{4(2)+1}}_{i=2} + \underbrace{\sqrt{4(\tfrac{5}{2})+1}}_{i=3} + \underbrace{\sqrt{4(3)+1}}_{i=4} + \underbrace{\sqrt{4(\tfrac{7}{2})+1}}_{i=5} + \underbrace{\sqrt{4(4)+1}}_{i=6}\right]$$

$$\approx 10.2820$$