Goal: Differentiate functions of the form $f(g(x))$. $(2x)^5$, $\sin(x^2)$, e^{-t}

Summary:

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$ $\frac{d}{dx}(f \circ g) = \frac{df}{dg} \cdot \frac{dg}{dx}$		
$[g(x)^p]' = p(g(x))^{p-1} \cdot g'(x)$	$(b^x)' = b^x \cdot \ln(b)$	$(\sin(\theta^\circ))' = \frac{\pi}{180}\cos(\theta^\circ)$

Theorem (Chain Rule) Let f and g be differentiable fcns.

Format I: $\frac{d}{dx}(f\circ g) = \frac{df}{dg} \cdot \frac{dg}{dx}$

Format II: $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

Easy (but incorrect) reason why

$$\frac{d}{dx}(f\circ g) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}\left(\frac{\Delta g}{\Delta g}\right) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta g} \cdot \frac{\Delta g}{\Delta x} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

Ex/ Compute $(\sin(x^2))'$

$g(x) =$ "inside fcn" $= x^2$

$f(g) =$ "outside fcn" $= \sin(g)$

Format I: $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx} = \frac{d}{dg}(\sin g) \cdot \frac{d}{dx}(x^2)$

$= \cos g \cdot 2x = 2x \cdot \cos(x^2)$

Format II: $[f(g(x))]' = [\sin(x^2)]' = \underbrace{[\sin(x^2)]'}_{\text{Differentiang w.r.t "}x^2\text{"}} \cdot \underbrace{[x^2]'}_{\text{w.r.t. } x}$

$= \cos(x^2) \cdot 2x$

Ex 2/ Compute $\left[\sin^2 x\right]' = \left[(\underbrace{\sin x})^2\right]'$ $\quad g(x) = \sin x$, $f(g) = g^2$

Format I: $\frac{d}{dx}\left[\sin^2 x\right] = \frac{df}{dg} \cdot \frac{dg}{dx} = \frac{d}{dg}(g^2) \cdot \frac{d}{dx}(\sin x)$

$$= \frac{d}{d(\sin x)}\left[(\sin x)^2\right] \cdot \frac{d}{dx}(\sin x) = 2\sin x \cdot \cos x$$

Format II: $\left[(\sin x)^2\right]' = f'(g) \cdot g' = \underbrace{\left[g^2\right]'}_{\text{w.r.t. } g} \cdot \underbrace{\left[\sin x\right]'}_{\text{w.r.t. } x}$

$$= 2g \cdot \cos x = 2\sin x \cos x$$

Ex 3/ Find y' if $y = e^{\underbrace{\cos x}_{g}}$ $\quad g(x) = \cos x$, $f(g) = e^g$

$$y' = \underbrace{\left[e^g\right]'}_{\text{w.r.t. } g} \cdot \underbrace{\left[\cos x\right]'}_{\text{w.r.t. } x} = e^g(-\sin x) = -e^g \sin x$$

$$= -e^{\cos x} \cdot \sin x$$

NOTE: It is common to write it as follows

$$y' = \underbrace{\left[e^{\cos x}\right]'}_{\text{"w.r.t. } \cos x\text{"}} \cdot \left[\cos x\right]' = e^{\cos x}(-\sin x)$$

Applications Of The Chain Rule

Ex 4/ Differentiate $y = (\underbrace{x^3 - 1}_{g})^{100}$

$f(g) = g^{100}$

$$y' = \underbrace{[(x^3-1)^{100}]'}_{\text{w.r.t. "}x^3-1\text{"}} \cdot \underbrace{[x^3-1]'}_{\text{w.r.t. } x} = 100(x^3-1)^{99}\cdot 3x^2$$

NOTE: $[g(x)]^{100} = 100\,[g(x)]^{99}\cdot g'(x)$

<u>Theorem (Generalized Power Rule)</u> If g is differentiable and p is any real number

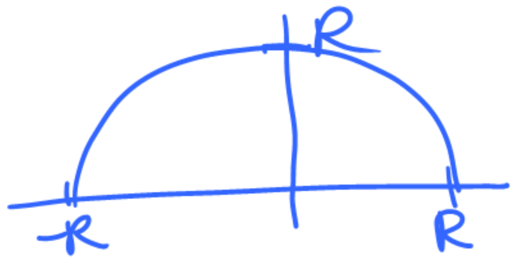
$$[(g(x))^p]' = p\,(g(x))^{p-1}\cdot g'(x)$$

$f(g) = g^p$

<u>Remark</u> If $g(x)=x$, then this is just the power rule

Ex 5/ Let $R>0$. The semicircle of radius R is given by

$$y = \sqrt{R^2-x^2}$$



What's y'?

$$y' = \left[\sqrt{R^2-x^2}\right]' = \left[(\underbrace{R^2-x^2}_{g})^{\frac{1}{2}}\right]' = \frac{1}{2}(R^2-x^2)^{-\frac{1}{2}}\cdot(-2x)$$

$f(g) = g^{1/2}$

$$= \frac{1}{2\sqrt{R^2-x^2}}(-2x) = -\frac{x}{\sqrt{R^2-x^2}}$$

Ex 6/ What's $\frac{d}{dx}(2^x)$?

$$[2^x]' = \left[e^{\ln(2^x)}\right]' = \left[e^{x\ln 2}\right]'$$

$g(x) = (\ln 2)x$
$f(g) = e^g$

$$= \underbrace{\left[e^g\right]'}_{\text{w.r.t. } g} \cdot \underbrace{\left[(\ln 2)x\right]}_{\text{w.r.t. } x} = e^g \cdot \ln(2) = e^{(\ln 2)x} \cdot \ln 2$$

$$= e^{\ln(2^x)} \cdot \ln(2) = 2^x \cdot \ln 2$$

In general, for any $b > 0$.

$$\frac{d}{dx}(b^x) = b^x \cdot \ln(b)$$

Ex 7/ Recall π radians $= 180° \Longrightarrow \dfrac{\frac{\pi}{180} \text{ radians}}{1 \text{ degree}}$

If an angle is measured in degrees

$$\sin(x°) = \sin\Big(\underbrace{\tfrac{\pi}{180}x}_{\text{measured in radians}}\Big)$$

What's $\frac{d}{dx}(\sin x)$ if x is measured in degrees?

$$[\sin x°]' = \Big[\sin\Big(\underbrace{\tfrac{\pi}{180}x}_{g}\Big)\Big]' = \cos\left(\tfrac{\pi}{180}x\right) \cdot \frac{\pi}{180} = \frac{\pi}{180}\cos\left(\tfrac{\pi}{180}x\right)$$

$f(g) = \sin(g)$

$$= \frac{\pi}{180}\cos(x°)$$

$$\boxed{\begin{array}{l}\sin(2x) \\ \cos(2x)\cdot 2\end{array}}$$

Ex 8/ If $h(x) = \sin\Big(\underbrace{\cos(\tan x)}_{g}\Big)$; $f(g) = \sin g$

$$h'(x) = \cos(\cos(\tan x)) \cdot \Big[\cos\big(\underbrace{\tan x}_{g}\big)\Big]'$$

$f(g) = \cos(g)$

$$= \cos(\cos(\tan x))\cdot[-\sin(\tan x)]\cdot[\tan x]'$$

$$= \underbrace{\cos(\cos(\tan x))}\cdot(-\sin(\tan x))\cdot \sec^2 x$$

$$f'(g(w(x))) \cdot g'(w(x)) \cdot w'(x)$$

<u>Ex 9 (Dampened Pendeleum)</u> A pendeleum's position is measured from its angle from its resting place

$$S(t) = e^{-t}\cos x$$

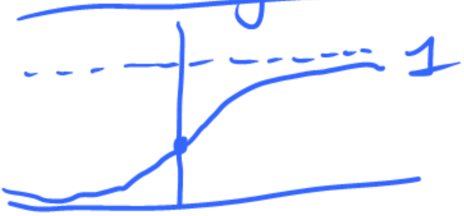
Find $v(t)$

$$v(t) = S'(t) = [e^{-t}]'\cos x + e^{-t}[\cos x]'$$

$$= -e^{-t}\cos x - e^{-t}\sin x$$

$$= -e^{-t}(\cos x + \sin x)$$

<u>Ex 10 (Logistics Curve)</u> Given $P(t) = \dfrac{1}{1+e^{-t}}$



$$P'(t) = \left[(1+e^{-t})^{-1}\right]' = -(1+e^{-t})^{-2}\cdot e^{-t}\cdot(-1)$$

$$= \frac{e^{-t}}{(1+e^{-t})^2}$$