Goal: Differentiate $\ln x$. Present the technique of Logarithmic Differentiation. Derive the power rule.
Summary:

$(\ln x)' = \frac{1}{x}$	$(\log_b x)' = \frac{1}{x \ln b}$
$(\ln f(x))' = \frac{f'}{f}$	$x^p = e^{p \ln x}$

Ex/ $\left(e^{\boxed{x^2}}\right)' = e^{x^2} \cdot (x^2)' = e^{x^2} \cdot 2x$

Ex/ $\left[e^{f(x)}\right]' = e^{f(x)} \cdot f'(x)$

Goal Use the previous example to find $\frac{d}{dx}(\ln x)$

$$f(x) = \ln x$$

$$\frac{d}{dx}\left(e^{f(x)}\right) = \frac{d}{dx}\left(e^{\ln x}\right) = \frac{d}{dx}(x)$$

$$e^{f(x)} \cdot f'(x) = 1$$

$$f'(x) = \frac{1}{e^{f(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Theorem $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Ex 1/ $[\ln(f(x))]' \stackrel{\text{Chain Rule}}{=} \frac{d}{d(f(x))}[\ln(f(x))] \cdot \frac{d}{dx}(f(x))$

$$= \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

Ex2/ $[x \cdot \ln(x)]' = [x]' \cdot \ln x + x[\ln x]'$

$$= \ln x + x \cdot \frac{1}{x} = \boxed{\ln x + 1}$$

Ex3/ $[\ln(\ln x)]' = \underbrace{[\ln(\boxed{\ln x})]'}_{\text{"w.r.t. } \ln x\text{"}} \cdot \underbrace{[\ln x]'}_{\text{w.r.t. } x}$

$= \frac{1}{\boxed{\ln x}} \cdot \frac{1}{x} = \frac{1}{x \ln x}$

Ex4/ $y = \ln(x^2-1)$

METHOD 1: $y' = [\ln(x^2-1)]' = \underbrace{[\ln(x^2-1)]'}_{\text{w.r.t "}x^2-1\text{"}} \cdot \underbrace{[x^2-1]'}_{\text{w.r.t. } x}$

$\ln(ab) = \ln a + \ln b$

$= \frac{1}{x^2-1} \cdot 2x = \boxed{\frac{2x}{x^2-1}}$

METHOD 2: $y = \ln(x^2-1) = \ln(x-1)(x+1) = \ln(x-1) + \ln(x+1)$

$y' = \frac{1}{x-1}(1) + \frac{1}{x+1}(1) = \boxed{\frac{1}{x-1} + \frac{1}{x+1}} = \frac{x+1+x-1}{(x-1)(x+1)}$

$= \frac{2x}{x^2-1}$

Ex5/ $y = \ln\left(\sqrt{\frac{x+1}{x^2+4}}\right) = \ln\left(\left[\frac{x+1}{x^2+4}\right]^{\frac{1}{2}}\right) = \frac{1}{2}\ln\left(\frac{x+1}{x^2+4}\right)$

$= \frac{1}{2}[\ln(x+1) - \ln(x^2+4)]$

$= \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x^2+4)$

$y' = \frac{1}{2} \cdot \frac{1}{x+1} \cdot 1 - \frac{1}{\cancel{2}} \cdot \frac{1}{x^2+4} \cdot \cancel{2}x$

$= \frac{1}{2(x+1)} - \frac{x}{x^2+4}$

Ex 6 $y = \log x = \log_{10} x \xlongequal[\text{Formula}]{\text{Change Of Base}} \dfrac{\ln(x)}{\ln(10)}$

$$y' = [\log_{10} x]' = \left[\frac{\ln(x)}{\ln(10)}\right]' = \frac{1}{\ln(10)}[\ln(x)]' = \frac{1}{x \cdot \ln(10)}$$

In general, for a base $b > 0$

$$\frac{d}{dx}(\log_b(x)) = \frac{1}{x \cdot \ln(b)}$$

Logarithmic Differentiation For $y = f(x)$

$[\ln(y)]' = \frac{1}{y} \cdot y' = \frac{y'}{y}$, we can use this to our advantage.

Ex 7 $(x^x)'$

$$y = x^x$$
$$\ln(y) = \ln(x^x)$$
$$\ln(y) = x\ln(x)$$
$$[\ln(y)]' = [x \ln x]'$$

$$\frac{y'}{y} = [x]' \ln x + x\,[\ln x]'$$
$$\frac{y'}{y} = \ln x + 1$$
$$y' = y[\ln x + 1]$$

Hence, $y' = x^x[\ln x + 1]$

Ex 8 $y = (2x+1)^5(x^4-3)^6$

$$\ln(y) = \ln\left[(2x+1)^5(x^4-3)^6\right] = \ln\left[(2x+1)^5\right] + \ln\left[(x^4-3)^6\right]$$

$$[\ln(y)]' = [5\ln(2x+1) + 6\ln(x^4-3)]'$$

$$\frac{y'}{y} = 5\cdot\frac{1}{2x+1}\cdot 2 + 6\cdot\frac{1}{x^4-3}(4x^3)$$

$$y' = y\left[\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right]$$

$$y = (2x+1)^5(x^4-3)^6\left[\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right]$$

$$= 10(2x+1)^4(x^4-3)^6 + 24x^3(x^4-3)^5(2x+1)^5$$

Ex 9/ $y = (\sin x)^x$

$$\ln y = \ln\left((\sin x)^x\right)$$

$$[\ln y]' = [x\ln(\sin x)]'$$

$$\frac{y'}{y} = [x]'\ln(\sin x) + x[\ln(\sin x)]'$$

$$\frac{y'}{y} = \ln(\sin x) + x\frac{1}{\sin x}\cdot\cos x$$

$$y' = y[\ln(\sin x) + x\cot x]$$

Hence, $y' = (\sin x)^x[\ln(\sin x) + x\cot x]$

Ex 10/ $y = \ln(\sec x + \tan x)$

$$y' = \frac{1}{\sec x + \tan x}(\sec x + \tan x)' = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$$

Ex 11 $y = \ln(x^4 \sin^2 x)$

$$y' = \frac{1}{x^4 \sin^2 x}\left[x^4 \sin^2 x\right]'$$

$$= \frac{1}{x^4 \sin^2 x}\left[(x^4)' \sin^2 x + x^4\left(\left[\sin x\right]^2\right)'\right]$$

$$= \frac{1}{x^4 \sin^2 x}\left[4x^3 \sin^2 x + x^4 (2\sin x)\cos x\right]$$

$$= \frac{4}{x} + \frac{2\cos x}{\sin x} = \frac{4}{x} + 2\cot x$$

Derivative Of The x^p

Recall $\frac{d}{dx}(x^p) = p x^{p-1}$. Why?

$$\frac{d}{dx}(x^p) = \frac{d}{dx}\left[e^{\ln(x^p)}\right] = \frac{d}{dx}\left[e^{p\ln x}\right]$$

$$= e^{p\ln x} \cdot (p\ln x)' = e^{p\ln x} \cdot \frac{p}{x}$$

$$= e^{\ln(x^p)} \cdot \frac{p}{x} = x^p \cdot \frac{p}{x} = p x^{p-1}$$