Goal: Differentiate $\ln x$. Present the technique of Logarithmic Differentiation. Derive the power rule.
Summary:

$(\ln x)' = \frac{1}{x}$	$(\log_b x)' = \frac{1}{x \ln b}$
$(\ln f(x))' = \frac{f'}{f}$	$x^p = e^{p \ln x}$

Ex/ $[e^{\boxed{x^2}}]' = e^{\boxed{x^2}} \cdot 2x$

In general,

$$[e^{f(x)}]' = e^{f(x)} \cdot f'(x)$$

Use this to find $\frac{d}{dx}(\ln x)$

$$f(x) = \ln x$$

$$\frac{d}{dx}\left(e^{f(x)}\right) = e^{\ln x} = \frac{d}{dx}(x)$$

$$e^{f(x)} \cdot f'(x) = 1$$

$$f'(x) = \frac{1}{e^{f(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

<u>Theorem</u> $\frac{d}{dx}(\ln x) = \frac{1}{x}$

What is $\frac{d}{dx}(\ln(f(x)))$?

$$[\ln(\boxed{f(x)})]' = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

Ex1/ $[x \ln x]' = [x]' \ln x + x[\ln x]'$

$$= 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

Ex2/ $[\ln(\ln x)]' = \underbrace{[\ln(\ln x)]'}_{\text{w.r.t. "}\ln x\text{"}} \cdot \underbrace{[\ln x]'}_{\text{w.r.t. } x}$

$$= \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x \ln x}$$

Ex3/ $[\ln(x^2-1)]'$

METHOD 1: $[\ln(\boxed{x^2-1})]' = \frac{1}{\boxed{}} \cdot \boxed{}'$

$$= \frac{1}{x^2-1} \cdot (x^2-1)' = \frac{2x}{x^2-1}$$

METHOD 2: $y = \ln(x^2-1) = \ln[(x-1)(x+1)] = \ln(x-1) + \ln(x+1)$

$$y' = \underbrace{\frac{1}{x-1}(1)} + \frac{1}{x+1}(1) = \boxed{\frac{1}{x-1} + \frac{1}{x+1}} = \frac{x+1+x-1}{(x-1)(x+1)}$$

$$= \boxed{\frac{2x}{x^2-1}}$$

Ex4/ $y = \ln\left(\sqrt{\frac{x+1}{x^2+4}}\right) = \ln\left(\left[\frac{x+1}{x^2+4}\right]^{\frac{1}{2}}\right) = \frac{1}{2}\ln\left[\frac{x+1}{x^2+4}\right]$

$$= \frac{1}{2}[\ln(x+1) - \ln(x^2+4)] = \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x^2+4)$$

$$y' = \frac{1}{2} \cdot \frac{1}{x+1}(1) - \frac{1}{\cancel{2}} \cdot \frac{1}{x^2+4} \cdot \cancel{2}x$$

$$= \boxed{\frac{1}{2x+2} - \frac{x}{x^2+4}}$$

Ex5/ $y = \log x = \log_{10} x \underset{\text{Formula}}{\overset{\text{Change of Base}}{=}} \frac{\ln x}{\ln 10}$

$$y' = [\log x]' = \left[\frac{\ln x}{\ln 10}\right]' = \frac{1}{\ln 10} \cdot [\ln x]' = \frac{1}{\ln 10} \cdot \frac{1}{x}$$

$$= \frac{1}{x \ln 10}$$

In general, for a base $b > 0$

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \cdot \ln b}$$

Logarithmic Differentiation Recall $y = f(x)$

$[\ln y]' = \frac{1}{y} \cdot y' = \frac{y'}{y}$. We can use this to our benefit.

Ex 6/ $[x^x]'$

$$y = x^x$$

$$\ln y = \ln(x^x)$$

$$[\ln y]' = [x \ln x]'$$

$$\frac{y'}{y} = [x]' \ln x + x[\ln x]'$$

$$\frac{y'}{y} = \ln x + 1$$

$$y' = y[\ln x + 1]$$

Hence, $y' = x^x[\ln x + 1]$

Ex 7/ $y = (2x+1)^5 (x^4 - 3)^6$

$$\ln(y) = \ln\left[(2x+1)^5 (x^4-3)^6\right] = \ln\left[(2x+1)^5\right] + \ln\left[(x^4-3)^6\right]$$

$$\ln(y) = 5\ln(2x+1) + 6\ln(x^4-3)$$

$$\frac{y'}{y} = 5\cdot\frac{1}{2x+1}(2) + 6\cdot\frac{1}{x^4-3}\cdot 4x^3$$

$$y' = y\left[\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right]$$

$$y' = (2x+1)^5(x^4-3)^6\left[\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right]$$

$$y' = 10(2x+1)^4(x^4-3)^6 + 24x^3(2x+1)^5(x^4-3)^5$$

Ex8/ $y = (\sin x)^x$

$$\ln(y) = \ln(\sin x)^x$$

$$[\ln(y)]' = [x\ln(\sin x)]'$$

$$\frac{y'}{y} = [x]'\ln(\sin x) + x[\ln(\sin x)]'$$

$$\frac{y'}{y} = \ln(\sin x) + x\frac{1}{\sin x}\cdot\cos x$$

$$y' = y[\ln(\sin x) + x\cot x]$$

Hence, $y' = (\sin x)^x[\ln(\sin x) + x\cot x]$

Ex9/ $y = \ln(\sec x + \tan x)$

$$y' = \frac{1}{\sec x + \tan x}[\sec x + \tan x]' = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$$

~~Ex 10~~ $y = \ln(x^4 \sin^2 x)$

$$y' = \frac{1}{x^4 \sin^2 x} \cdot \left[x^4 (\sin x)^2\right]'$$

$$= \frac{1}{x^4 \sin^2 x}\left[(x^4)' \sin^2 x + x^4 \left((\sin x)^2\right)'\right]$$

$$= \frac{1}{x^4 \sin^2 x}\left[4x^3 \sin^2 x + x^4 \, 2\sin x \cos x\right]$$

$$= \frac{4}{x} + \frac{2\cos x}{\sin x} = \frac{4}{x} + 2\cot x$$

Derivative of x^p Recall $\frac{d}{dx}(x^p) = p x^{p-1}$

Why?

$$\frac{d}{dx}(x^p) = \frac{d}{dx}\left(e^{\ln(x^p)}\right) = \frac{d}{dx}\left(e^{p \ln x}\right)$$

$$= e^{p\ln x} \cdot (p \ln x)' = e^{p \ln x} \cdot \frac{p}{x}$$

$$= \left[e^{\ln(x^p)}\right] \cdot \frac{p}{x} = x^p \cdot \frac{p}{x} = p x^{p-1}$$