MA 16010: Applied Calculus I

Lecture 14: Related Rates (Geometric Relations)

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Sections Covered: 3.1 (Up to the Ladder Problem)

Introduction

★ Assume all variables are functions of time

A circle's area and radius are related by the equation:

$$A = \pi r^2 \qquad A_{(t)} = \pi [r_{(t)}]^2$$

If A are r are changing as time advances, is there any relation between $\frac{dA}{dt}$ and $\frac{dr}{dt}$?

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\pi [r_{(t)}]^2\right)$$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

Plugging in Values

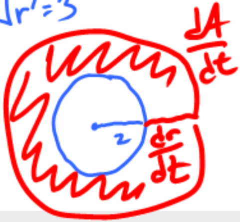
$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

Problem 1

In the previous example, if $r = 2$ and $\frac{dr}{dt} = 3$, then what is the value of $\frac{dA}{dt}$? Interpret.

We know: $r = 2$, $\frac{dr}{dt} = 3$

$$\left.\frac{dA}{dt}\right|_{\substack{r=2 \\ r'=3}} = 2\pi(2)(3) = 12\pi$$



When the radius is 2, the area is increasing at a rate of $12\pi \frac{\text{units}^2}{\text{sec}}$

Plugging in Values (cont.)

$$\boxed{\frac{dA}{dt} = 2\pi r \frac{dr}{dt}}$$

Problem 2

In the previous example, if $r = 1$ *and* $\frac{dA}{dt} = 2\pi$*, then what is the value of* $\frac{dr}{dt}$*?*

$$\frac{dr}{dt} = \frac{1}{2\pi r} \frac{dA}{dt}$$

$$\left.\frac{dr}{dt}\right]_{\substack{r=1 \\ A'=2\pi}} = \frac{1}{2\pi(1)}(2\pi) = 1$$

Applying the circle example

Problem 3

The radius of a circle r is increasing at a constant rate 3 cm/min.

(1) Find the rate of change of the area of the circle (A) when the radius is 5cm.

Know:

$\frac{dr}{dt} = 3\frac{cm}{min}$; $r = 5cm$

Need to Know (NTK):

$\frac{dA}{dt}$

$A = \pi r^2$

$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$

$\pi[r(t)]^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$\left.\frac{dA}{dt}\right]_{r'=3}^{r=5} = 2\pi(5)(3)$

$= 30\pi \frac{cm^2}{min}$

When the radius is 5cm the area is increasing at a rate of $30\pi \frac{cm^2}{min}$

Applying the circle example (cont.)

(2) Find the rate of change of the circumference of the circle (C) when the radius is 5cm

Know:
$r = 5$
$\frac{dr}{dt} = 3$

Need to Know:
$\frac{dC}{dt}$ when $r = 5$

$$C = 2\pi r$$

$$\frac{d}{dt}(C) = \frac{d}{dt}(2\pi r)$$

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$$

$$\left.\frac{dC}{dt}\right]_{\substack{r=5 \\ r'=3}} = 2\pi(3)$$

$$= 6\pi \ \frac{\text{cm}}{\text{min}}$$

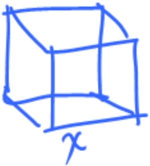
When the radius is 5, the circumference is increasing at a rate of $6\pi \frac{\text{cm}}{\text{s}}$

Rectangular Prisms

Problem 4

The edges of a cube are shrinking at a rate of 10 *cm/s.*

(1) How fast is the volume (V) shrinking when each side length is 9cm long?



Know:

$\frac{dx}{dt} = -10 \frac{cm}{s}$

$x = 9$

$V = x^3$

Need to know:

$\frac{dV}{dt}$ when $x = 9$

$V = x^3$

$\frac{d}{dt}(V) = \frac{d}{dt}(x^3)$

$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

$\left.\frac{dV}{dt}\right]_{\substack{x=9 \\ x'=-10}} = 3(9)^2(-10)$

$\left.\frac{dV}{dt}\right]_{\substack{x=9 \\ x'=-10}} = -2430 \frac{cm^3}{s}$

When the side lengths are 9cm the volume is decreasing by 2430 cm³/s

Rectangular Prisms (cont.)

(2) How fast is the surface area (A) shrinking when each side length is 9cm long?

Know:
$x = 9$
$dx/dt = -10$

Need to know:
$\frac{dA}{dt}$ when $x = 9$

Surface Area of a cube
$A = 6x^2$

$\frac{d}{dt}(A) = \frac{d}{dt}(6x^2)$

$\frac{dA}{dt} = 12x\frac{dx}{dt}$

$\left.\frac{dA}{dt}\right]_{\substack{x=9 \\ x'=-10}} = 12(9)(-10)$

$= -1080\frac{cm^2}{s}$

When the side length is 9cm, the surface area is decreasing at a rate of $1{,}080\frac{cm^2}{s}$

Spheres

Problem 5

A balloon is (roughly) a sphere. The balloon deflates and its radius decreases at a rate of 2 cm/s.

(1) How fast is the volume (V) shrinking when the radius is 5cm long?

Know:
$\frac{dr}{dt} = -2 \frac{cm}{s}$
$r = 5$ cm

NTK:
$\frac{dV}{dt}$

Volume of a sphere:
$V = \frac{4\pi}{3} r^3$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\left.\frac{dV}{dt}\right]_{\substack{r=5 \\ r'=-2}} = 4\pi (5)^2 (-2)$$

$$\left.\frac{dV}{dt}\right]_{\substack{r=5 \\ r'=-2}} = -200\pi \frac{cm^3}{s}$$

When the radius is 5 cm, the volume is decreasing at a rate of $200\pi \frac{cm^3}{s}$

Spheres (cont.)

(2) How fast is the surface area (A) shrinking when the radius is 5cm long?

K:	NTK:	
$r=5$	$\frac{dA}{dt}$	$A=4\pi r^2$
$r'=-2$		

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$$

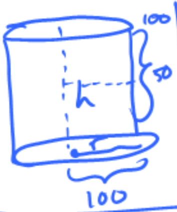
$$\left.\frac{dA}{dt}\right]_{\substack{r=5\\ r'=-2}} = 8\pi(5)(-2) = -80\pi \ \frac{cm^2}{s}$$

When the radius is 5 cm, the surface area of the sphere is decreasing by $80\pi \ \frac{cm^2}{s}$.

Cylinders

Problem 6

A cylindrical tank with a radius and height of 100 *cm stands upright. Water is being drained at a rate of* $7\,\text{cm}^3/s$*. How fast is the water level changing when the tank is half empty.*



K:
$r = 100$
$\frac{dV}{dt} = -7\frac{\text{cm}^3}{s}$

NTK:
$\frac{dh}{dt}$

Volume of a Cylinder:
$V = \pi r^2 h$

$$\frac{dV}{dt} = \Big(2\pi r \underbrace{\frac{dr}{dt}}_{=0}\Big)h + \pi r^2 \frac{dh}{dt} \Rightarrow \frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

$$-7 = \pi(100)^2 \frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} = -2.2\frac{\text{mm}}{s}}$$

Cones

Problem 7

Sand pours onto a surface at 15cm³/s, forming a conical pile with a base diameter that is always equal to the pile's altitude. How fast is the altitude of the pile increasing when the pile is 8cm high?

NTK: $\frac{dh}{dt}$

K: $\frac{dV}{dt} = 15$, $h = 8$, $2r = h$

Formula:

$$V = \frac{\pi}{3} r^2 h$$

$$V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3$$

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{15}{16\pi} \ \frac{cm}{s}$$