MA 16010: Applied Calculus I

Lecture 14: Related Rates (Geometric Relations)

Zachariah Pence

Purdue University

Sections Covered: 3.1 (Up to the Ladder Problem)

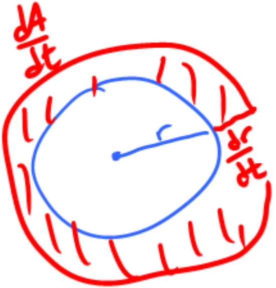
Introduction

★ Assume all variables are functions of time

A circle's area and radius are related by the equation:

$$A = \pi r^2 \qquad A(t) = \pi [r(t)]^2$$

If A are r are changing as time advances, is there any relation between $\frac{dA}{dt}$ and $\frac{dr}{dt}$?



$$A = \pi r^2$$

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\pi [r(t)]^2\right)$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Plugging in Values

Problem 1

In the previous example, if $r = 2$ and $\frac{dr}{dt} = 3$, then what is the value of $\frac{dA}{dt}$? Interpret.

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\left.\frac{dA}{dt}\right]_{\substack{r=2 \\ r'=3}} = 2\pi(2)(3) = 12\pi$$

When $r=2$, the area is increasing at a rate of 12π units^2/sec

Plugging in Values (cont.)

Problem 2

In the previous example, if $r = 1$ and $\frac{dA}{dt} = 2\pi$, then what is the value of $\frac{dr}{dt}$?

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{2\pi r} \frac{dA}{dt}$$

$$\left.\frac{dr}{dt}\right]_{\substack{r=1\\ A'=2\pi}} = \frac{1}{2\pi(1)}(2\pi) = 1$$

Applying the circle example

Problem 3

The radius of a circle r is increasing at a constant rate 3 cm/min.

(1) Find the rate of change of the area of the circle (A) when the radius is 5cm.

Know:
$\frac{dr}{dt} = 3$
$r = 5$

Need to Know (NTK):
$\frac{dA}{dt}$

Formula:
$A = \pi r^2$

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\left.\frac{dA}{dt}\right]_{\substack{r=5 \\ r'=3}} = 2\pi(5)(3) = 30\pi \frac{cm^2}{min}$$

When the radius is 5cm, the area of the circle is increasing at a rate of $30\pi \frac{cm^2}{min}$.

Applying the circle example (cont.)

(2) Find the rate of change of the circumference of the circle (C) when the radius is 5cm.

K: $\frac{dr}{dt} = 3$, $r = 5$ | NTK: $\frac{dC}{dt}$ | Formula: $C = 2\pi r$

$$C = 2\pi r$$
$$\frac{d}{dt}(C) = \frac{d}{dt}(2\pi r)$$
$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$$

$$\left.\frac{dC}{dt}\right]_{\substack{r=5 \\ r'=3}} = 2\pi(3) = 6\pi \frac{cm}{min}$$

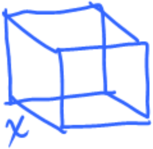
When the radius is 5 cm, the circumference is increasing at a rate of $6\pi \frac{cm}{min}$

Rectangular Prisms

Problem 4

The edges of a cube are shrinking at a rate of 10 *cm/s.*

(1) How fast is the volume (V) shrinking when each side length is 9cm long?



K: $\frac{dx}{dt} = -10$

$x = 9\text{cm}$

NTK: $\frac{dV}{dt}$ when $x = 9$

Formula: $V = x^3$

$$\frac{d}{dt}(V) = \frac{d}{dt}(x^3)$$

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$\left.\frac{dV}{dt}\right]_{\substack{x=9 \\ x'=-10}} = 3(9)^2(-10) = -2430 \ \frac{\text{cm}^3}{\text{s}}$$

When the side lengths are 9cm, the volume is decreasing at a rate of $2430 \ \frac{\text{cm}^3}{\text{s}}$.

Rectangular Prisms (cont.)

(2) How fast is the surface area (A) shrinking when each side length is 9cm long?

K:
$x = 9$
$\frac{dx}{dt} = -10$

NTK:
$\frac{dA}{dt}$

Formula:
$A = 6x^2$

$$\frac{d}{dt}(A) = \frac{d}{dt}(6x^2)$$

$$\frac{dA}{dt} = 12x\frac{dx}{dt}$$

$$\left.\frac{dA}{dt}\right]_{\substack{x=9 \\ x'=-10}} = 12(9)(-10) = -1{,}080\,\frac{cm^2}{s}$$

When the side lengths are 9 cm, the surface area of the cube is decreasing at a rate of 1,080 $\frac{cm^2}{s}$.

Spheres

Problem 5

A balloon is (roughly) a sphere. The balloon deflates and its radius decreases at a rate of 2 cm/s.

(1) How fast is the volume (V) shrinking when the radius is 5cm long?

K: $\frac{dr}{dt} = -2$, $r = 5$

NTK: $\frac{dV}{dt}$

Formula: $V = \frac{4\pi}{3} r^3$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\left.\frac{dV}{dt}\right]_{\substack{r=5 \\ r'=-2}} = 4\pi(5)^2(-2)$$

$$\left.\frac{dV}{dt}\right]_{\substack{r=5 \\ r'=-2}} = -200\pi \ \text{cm}^3/\text{s}$$

When the radius is 5 cm, the volume is decreasing by 200π cm^3/s

Spheres (cont.)

(2) How fast is the surface area (A) shrinking when the radius is 5cm long?

K: $r=5$, $\frac{dr}{dt}=-2$ | NTK: $\frac{dA}{dt}$ | Formula: $A = 4\pi r^2$

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$$

$$\left.\frac{dA}{dt}\right]_{\substack{r=5 \\ r'=-2}} = 8\pi(5)(-2)$$

$$= -80\pi \ \frac{cm^2}{s}$$

When the radius of the circle is 5 cm, its surface area is decreasing at a rate of 80π cm^2/s.

Problem 6

A cylindrical tank with a radius and height of 100 *cm stands upright. Water is being drained at a rate of* $7\,cm^3/s$. *How fast is the water level changing when the tank is half empty.*

NTK:
"Water Level"
$\Downarrow$
$\frac{dh}{dt}$

K:
$r = 100cm$
$h = 50cm$
$\frac{dV}{dt} = -7$

Formula:
$V = (\pi r^2)h$

$$\frac{dV}{dt} = \left(2\pi r \underbrace{\frac{dr}{dt}}_{=0}\right)h + \pi r^2 \frac{dh}{dt}$$

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

$$-7 = \pi (100)^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{-7}{10000\pi} \frac{cm}{s}$$

$$\Rightarrow \frac{dh}{dt} \approx -2.2 \frac{mm}{s}$$

Cones

Problem 7

Sand pours onto a surface at $15 cm^3/s$, forming a conical pile with a base diameter that is always equal to the pile's altitude. How fast is the altitude of the pile increasing when the pile is 8cm high?

NTK: $\frac{dh}{dt}$

K: $\frac{dV}{dt} = 15$, $h = 8$, $2r = h$

Formula:

$$V = \frac{\pi}{3} r^2 h$$

$$V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3$$

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{15}{16\pi} \frac{cm}{s}$$