Lecture 15: Related Rates (Trigonometric Relations)

Assume all variables are functions of time.

1 Pythagorean Relations

Problem 1. *Suppose $x = x(t)$, $y = y(t)$, and $z = z(t)$ are related by the Pythagorean Theorem. So,*

$$x^2 + y^2 = z^2$$

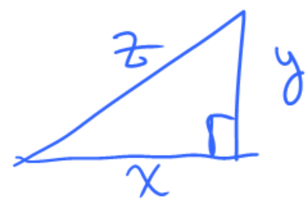
How are $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$ related? In particular, what happens when $\frac{dz}{dt} = 0$?

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(z^2)$$

$$\frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2} = \frac{2z\frac{dz}{dt}}{2}$$

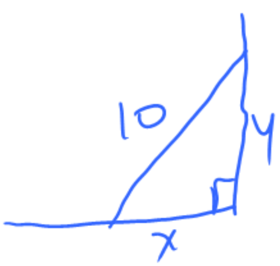
$$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$

In particular, when $\frac{dz}{dt} = 0$



$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Problem 2. *A ladder 10m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6m from the wall?*



NTK: $\frac{dy}{dt}$ and y when $x = 6$

K: $\frac{dx}{dt} = 1$

Formula: $x^2 + y^2 = 10^2$

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(10^2)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y}\cdot\frac{dx}{dt}$$

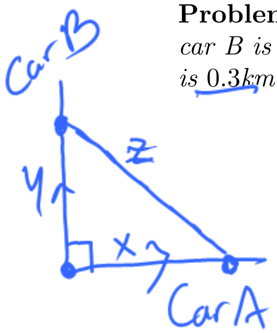
Find y: $x^2+y^2=10^2 \xrightarrow{y>0} y = \sqrt{100-x^2}$

When $x=6$, $y = \sqrt{100-36} = \sqrt{64} = 8$

$$\left.\frac{dy}{dt}\right|_{\substack{x=6\\x'=1}} = -\frac{6}{8}(1) = -\frac{3}{4}\ \frac{m}{s}$$

The top of the ladder is falling at a rate of $\frac{3}{4}$ m/s.

Problem 3. *Two cars leave from the same point at the same time. Car A is traveling* ~~*east*~~ *at* 50km/h *and car B is traveling north at* 60km/h*. At what rate is the distance between the two cars changing when car A is* 0.3km *and car B is* 0.4km *from their starting point?*



NTK: $\frac{dz}{dt}$ when $x = 0.3$ and $y = 0.4$

K: $\frac{dx}{dt} = 50$, $\frac{dy}{dt} = 60$

Formula: $x^2 + y^2 = z^2$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

Find z: $z = \sqrt{x^2+y^2}$

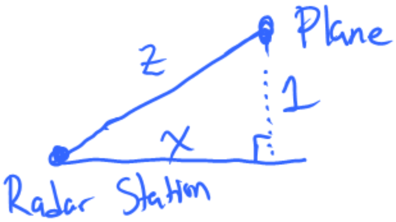
When $x = 0.3$, $y = 0.4$,

$z = \sqrt{0.3^2 + 0.4^2} = 0.5$

$$\left.\frac{dz}{dt}\right]_{(\text{Values})} = \frac{1}{0.5}\left(0.3(50) + 0.4(60)\right) = 78\ \frac{km}{h}$$

The distance betweens Cars A and B is increasing at a rate of $78\ \frac{km}{h}$

Problem 4. *A plane flying horizontally at an altitude of* 1 *km and a speed of* 500km/h *passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is* 2km *away from the station.*



NTK: $\frac{dz}{dt}$ when $z = 2$

K: $\frac{dx}{dt} = 500$

Formula: $x^2 + 1^2 = z^2$

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

$$\frac{x\frac{dx}{dt}}{z} = \frac{z\frac{dz}{dt}}{z}$$

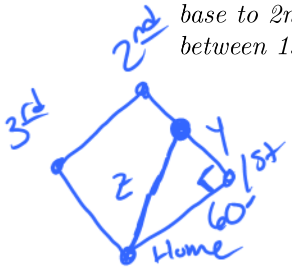
$$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt}$$

Find x: $x = \sqrt{z^2 - 1} \Rightarrow$ when $z = 2$, $x = \sqrt{2^2 - 1} = \sqrt{3}$

$$\left.\frac{dz}{dt}\right]_{\substack{z=2\\x'=500}} = \frac{\sqrt{3}}{2}(500) = 250\sqrt{3}\ km/h$$

The distance between the plane and the radar station is increasing at a rate of $250\sqrt{3}$ km/h.

Problem 5. *A Little League baseball diamond is a square with 60ft on each side. A player runs from 1st base to 2nd base at 12ft/s. At what rate is the player's distance from home plate increasing if he is halfway between 1st and 2nd base?*



NTK: $\frac{dz}{dt}$ when $y = 30$

K: $\frac{dy}{dt} = 12 \frac{ft}{s}$

Formula: $60^2 + y^2 = z^2$

$$0 + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$\frac{y\frac{dy}{dt}}{z} = \frac{z\frac{dz}{dt}}{z}$$

$$\frac{dz}{dt} = \frac{y}{z}\frac{dy}{dt}$$

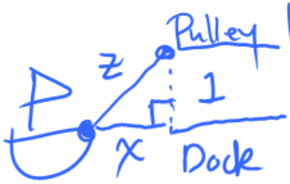
Find z: $z = \sqrt{60^2 + y^2}$; when $y = 30$

$z = \sqrt{60^2 + 30^2} = 30\sqrt{5}$

$$\left.\frac{dz}{dt}\right]_{\substack{y=30 \\ y'=12}} = \frac{30}{30\sqrt{5}}(12) = \frac{12}{\sqrt{5}} \frac{ft}{s}$$

The distance between the player and home plate increases at a rate of $\frac{12}{\sqrt{5}} \frac{ft}{s}$.

Problem 6. *A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1m higher than the bow of the boat. If the rope is pulled in at a rate of 1m/s, how fast is the boat approaching the dock when it is 8m from the dock?*



NTK: $\frac{dx}{dt}$ when $x = 8$

K: $\frac{dz}{dt} = -1$

Formula: $x^2 + 1^2 = z^2$

$$\frac{x\frac{dx}{dt}}{x} = \frac{z\frac{dz}{dt}}{x}$$

$$\frac{dx}{dt} = \frac{z}{x}\frac{dz}{dt}$$

Find z: $z = \sqrt{x^2 + 1}$, when $x = 8$

$z = \sqrt{8^2 + 1} = \sqrt{65}$

$$\left.\frac{dx}{dt}\right]_{\substack{x=8 \\ z'=-1}} = \frac{\sqrt{65}}{8}(-1) = -\frac{\sqrt{65}}{8} \frac{m}{s}$$

The boat is approaching the dock at a rate of $\frac{\sqrt{65}}{8} \frac{m}{s}$

2 Trigonometric Relations

Problem 7. *A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15ft from the point on the path closest to the searchlight?*

NTK: $\frac{d\theta}{dt}$ when $x=15$

K: $\frac{dx}{dt} = 4\frac{ft}{s}$

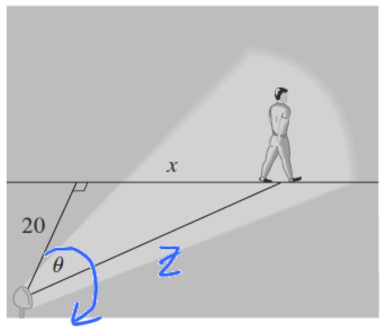


Figure 1: §3.10, Fig. 5, Stewart Calculus 5th Edition

Formula: $\frac{x}{20} = \tan\theta$

$x = 20\tan\theta$

$\frac{d}{dt}(x) = \frac{d}{dt}(20\tan\theta)$

$$\frac{dx}{dt} = 20\sec^2\theta \frac{d\theta}{dt}$$

$$\frac{dx}{dt} \Big/ 20\sec^2\theta = \frac{20\sec^2\theta}{20\sec^2\theta}\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{20}\cos^2\theta \frac{dx}{dt}$$

Find $\cos\theta$:

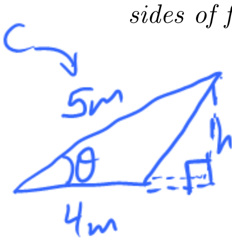
When $x=15$, $z = \sqrt{20^2+15^2} = \sqrt{625} = 25$

$$\cos\theta = \frac{20}{z} = \frac{20}{25} = \frac{4}{5}$$

$$\frac{d\theta}{dt}\Big]_{\cos\theta=\frac{4}{5}} = \frac{1}{20}\left(\frac{4}{5}\right)^2(4) = \frac{16}{125}\ \frac{rad}{s}$$

The searchlight is rotating clockwise at a rate of $\frac{16}{125}\frac{rad}{s}$

Problem 8. *Two sides of a triangle are 4m and 5m in length and the angle between them is increasing at a rate of* 0.06*rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is* $\frac{\pi}{3}$.



$\frac{h}{5} = \sin\theta$

NTK: A is area, $\frac{dA}{dt}$ when $\theta = \frac{\pi}{3}$

K: $\frac{d\theta}{dt} = 0.06\frac{rad}{s}$

Formula:

$A = \frac{1}{2}bh$

$= \frac{1}{2}b(c\sin\theta)$

$A = \frac{1}{2}(4)(5)\sin\theta = 10\sin\theta$

$$\frac{dA}{dt} = 10\cos\theta\frac{d\theta}{dt}$$

$$\frac{dA}{dt}\Big]_{\substack{\theta=\frac{\pi}{3} \\ \theta'=0.06}} = 10\Big[\underbrace{\cos\frac{\pi}{3}}_{1/2}\Big]\frac{6}{100} = \frac{30}{100} = \frac{3}{10}\frac{m^2}{s}$$

The area of the triangle is increasing at a rate of $3/10\ m^2/s$.