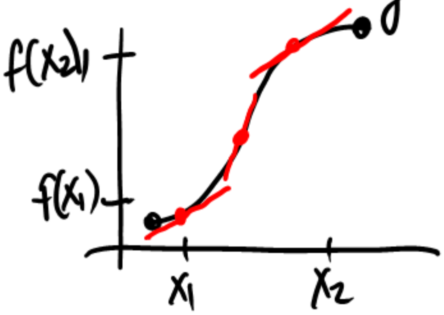
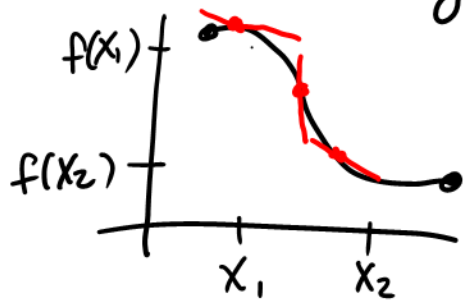
Lecture 17: I/D and 1st Derivative Test

Goal: Use the 1st derivative to determine properties of the original function.

Increasing vs Decreasing



If $x_1 < x_2$, $f(x_1) < f(x_2)$



If $x_1 < x_2$, $f(x_2) < f(x_1)$

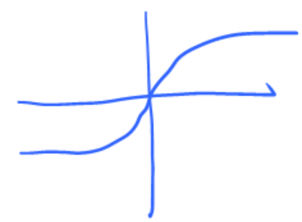
Theorem (Inc./Dec. Test) Let f be differentiable on an open interval I.

1. If $f'(x) > 0$ for any x in I, f is increasing on I
2. If $f'(x) < 0$ for any x in I, f is decreasing on I

Ex1/ When is $f(x) = \frac{e^x}{1+e^x}$ increasing and decreasing

Step 1 Determine the critical numbers of f

$$f'(x) = \frac{e^x(1+e^x) - e^x(e^x)}{(1+e^x)^2} = \frac{\overset{>0}{e^x}}{\underset{>0}{(1+e^x)^2}} \overset{\text{set}}{=} 0$$



There are none $\Rightarrow$ f' is always positive or negative

Step 2 Plug in a test point

$f'(0) = \frac{1}{1} = 1 \Rightarrow f'$ is increasing always

Ex2/ When does the fcn $\frac{x^2}{1+x^2}$ increase and decrease

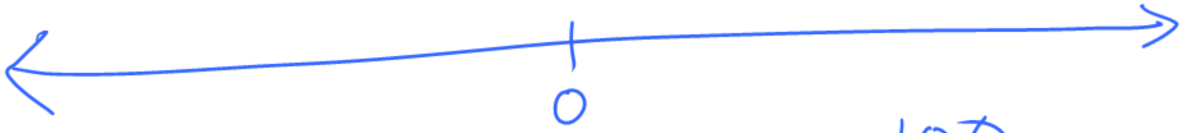
Step 1 Find Critical Numbers

$$f'(x) = \frac{2x(1+x^2) - x^2(2x)}{(1+x^2)^2} = \frac{2x}{\underbrace{(1+x^2)^2}_{>0}} \overset{\text{set}}{=} 0$$

$2x = 0 \implies \boxed{x=0}$

Step 2 Use c.n.s. to divide x-axis to determine the sign of f'

$$\boxed{f'(x) = \frac{2x}{(1+x^2)^2}}$$



Test Point f'	-100	100
Sign of $2x$	$-$	$+$
Sign of $(1+x^2)^2$	$+$	$+$
Sign of f'	$-$	$+$

Local Minimum at $x=0$

Conclusion: f is decreasing on $(-\infty, 0)$ and f is increasing on $(0, \infty)$

Ex3/ Repeat for $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

Step 1 Determine our critical numbers

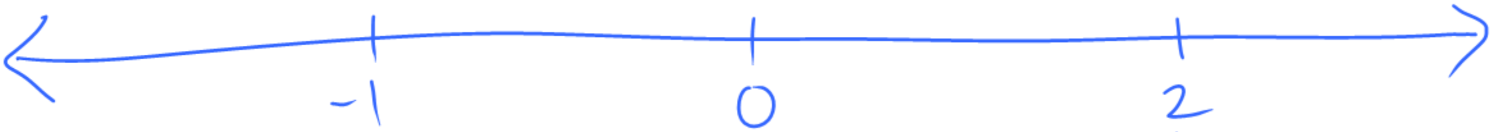
$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1) \overset{\text{set}}{=} 0$$

Either $12x = 0$ OR $x - 2 = 0$ OR $x + 1 = 0$

$\boxed{x = 0}$ $\boxed{x = 2}$ $\boxed{x = -1}$

$12x(x-2)(x+1)$

Step 2 Divide x-axis into region, determine sign of f'

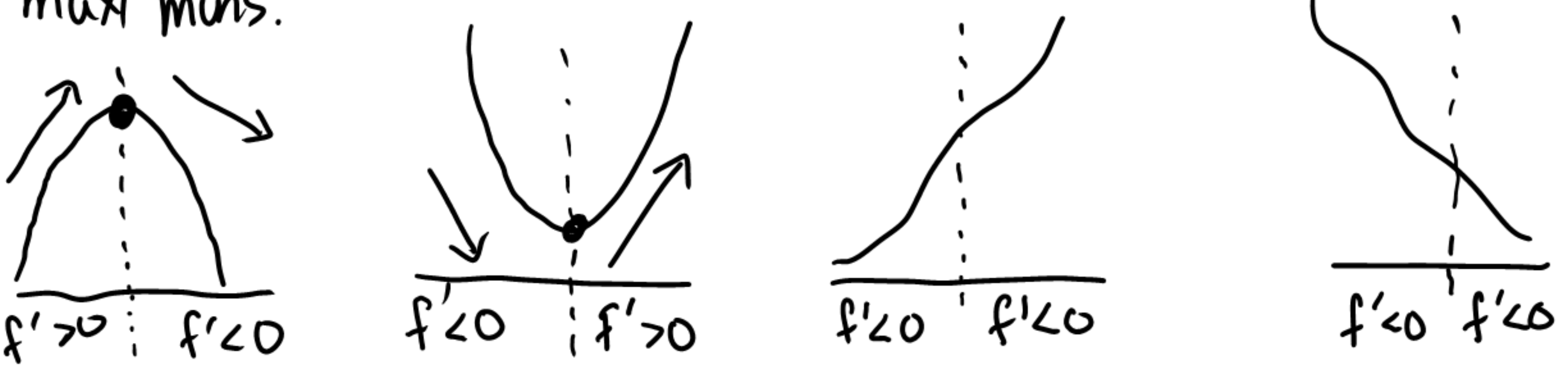


Test Point	-100		$-\frac{1}{2}$		1		100
Sign of $2x$	$-$		$-$		$+$		$+$
Sign of $x-2$	$-$		$-$		$-$		$+$
Sign $x+1$	$-$		$+$		$+$		$+$
Sign of f'	$-$	min	$+$	max	$-$	min	$+$
	$(-\infty, -1)$		$(-1, 0)$		$(0, 2)$		$(2, \infty)$

Conclusion f is increasing on $(-1, 0) \cup (2, \infty)$

f is decreasing on $(-\infty, -1) \cup (0, 2)$

The I/D Test gives us a way to detect rel. max/ mins.



Thm (1st Derivative Test) Let c be a critical number of a differentiable function f

① If f' switches from being positive to negative at c, then f has a rel. max at c

② If f' switches from being negative to positive at c, then f has a rel. min at c

③ If f' doesn't change sign at c, f has neither a rel. max nor min at c.

Ex4/ Go back to determine the locations of rel. max/mins of the previous exs.

Ex1/ $\frac{e^x}{1+e^x}$; none

Ex2/ Rel. min. at $x=0$

Ex3/ Rel. min at $x=-1,2$. Rel. max at $x=0$

Ex5/ The critical values of $g(x)=x+2\sin x$ are $x=\frac{2\pi}{3},\frac{4\pi}{3}$, determine the locations of the rel. max/mins on the interval $(0,2\pi)$

$$g'(x)=1+2\cos x$$

Test Point	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
Sign of g'	$g'(\frac{\pi}{2}) = 1+2\cdot 0 = 1$ $+$	$g'(\pi) = 1+2(-1) = -1$ $-$	$g'(\frac{3\pi}{2}) = 1+2\cdot 0 = 1$ $+$

Conclusion: g has a relative max at $x = \frac{2\pi}{3}$ while
g has a relative minimum at $x = \frac{4\pi}{3}$

Ex 6/ Repeat for $f(x) = x^4 - 6x^2$

Step 1 Find crit. nums.

$$f'(x) = 4x^3 - 12x = 4x(x^2 - 3) \overset{\text{set}}{=} 0$$

$4x = 0$ OR $x^2 - 3 = 0$

$\boxed{x = 0}$ $\qquad x = \pm\sqrt{3}$

Step 2 Make sign chart for $4x(x^2-3)$

$-\sqrt{3}$ $\qquad 0 \qquad$ $\sqrt{3}$

Test Points:	-100	-1	1	100
$4x$:	$-$	$-$	$+$	$+$
x^2-3:	$+$	$(-1)^2 - 3 = -2$ $-$	$1^2 - 3 = -2$ $-$	$+$
f':	$-$	$+$	$-$	$+$

Res : Dec | Inc | Dec | Inc

Min Max Min