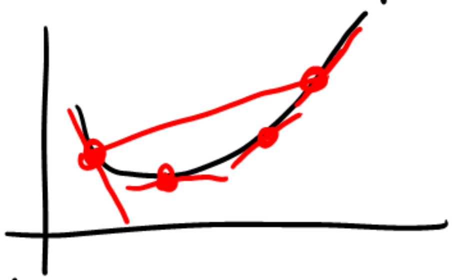
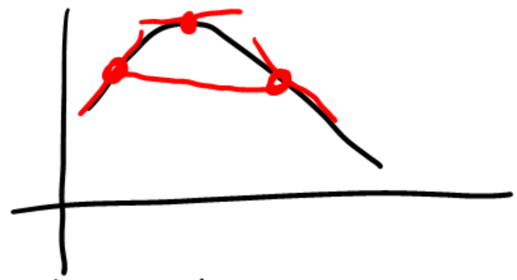
Goal: Use the 2nd derivative to determine properties of the original function.

Concave Up (CU) [Convex] Concave Down (CD)





f' is increasing $[f']' = f'' > 0$

f' is decreasing, $f'' < 0$

Theorem (Concavity Test) Let f be a function where f'' exists on an open interval I

1. If $f''(x) > 0$ for all x in I, f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, f is concave downward on I.

Ex) Discuss the shape of $f(x) = x^3 - 12x + 1$

Step Find when f' and f'' are 0

$$f'(x) = 3x^2 - 12 = 3(x-2)(x+2) \overset{\text{set}}{=} 0$$

$$\Rightarrow \boxed{x = -2, 2}$$

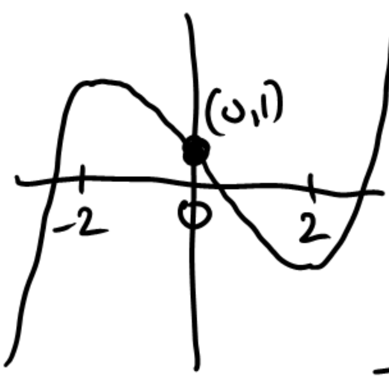
$$f''(x) = 6x \overset{\text{set}}{=} 0 \Rightarrow \boxed{x = 0}$$

Test Points | -100 | -1 | 1 | 100

Sign of $3(x-2)(x+2)$	+	−	−	+
Sign of $6x$	−	−	+	+
Results of I/D Test	Inc	Dec	Dec	Inc
Results of C Test	CD	CD	CU	CU

Conclusion: f is inc and CD on $(-\infty,-2)$
f is dec and CD on $(-2,0)$
f is dec and CU on $(0,2)$
f is inc and CU on $(2,\infty)$

Remark There is a way to memorize the C Test



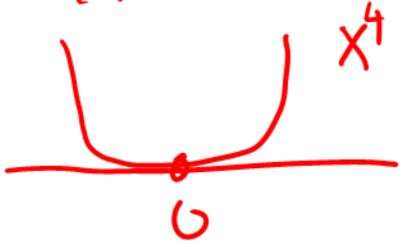
Def If f changes concavity at a point p, then p is called an inflection point of f.

Theorem If p is an inflection point, either $f''(p) = 0$ or undefined

Why? Inflection points are crit. nums. of f'

$[f']' = 0$ or undefined

Remark The reverse is not true



Ex 3/ Determine the location of the inflection points for $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$

Idea Set $f'' = 0$

$$f'(x) = 12x^3 - 12x^2 - 12x + 12$$

$$f''(x) = 36x^2 - 24x - 12$$

$$= 12(3x^2 - 2x - 1) = 12(x-1)(3x+1)$$

$$\underline{\underline{\text{set}}}\ 0$$

Either $\cancel{12 = 0}$ OR $x - 1 = 0$ OR $3x + 1 = 0$

$\boxed{x = 1}$ $\boxed{x = -\frac{1}{3}}$

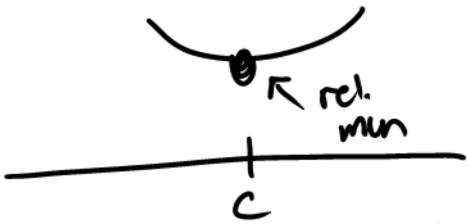
Step 2 Make sign chart for f''

Test Point	-2	0	2
Sign of 12:	+	+	+
Sign of $x-1$:	−	−	+
Sign of $3x+1$	−	+	+
Sign of f''	+	−	+
Results of CTest	CU	CD	CU

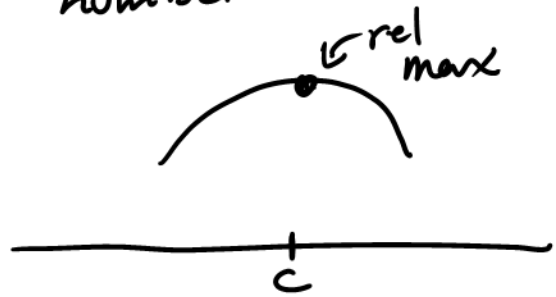
Conclusion: Both $x = -\frac{1}{3}$ and $x = 1$ are inflection points

The 2^{nd} Derivative can be used to detect rel. max/min

Let c be a critical number



$f''(c) > 0 \Rightarrow f$ is CU near c



$f''(c) < 0 \Rightarrow f$ is CD near c

Theorem (2nd Derivative Test) Let f'' be continuous near c and $f'(c) = 0$

1. If $f''(c) > 0$, then f has a relative min at c.
2. If $f''(c) < 0$, then f has a relative max at c.
3. If $f''(c) = 0$, the test is inconclusive (pick another test)

Ex3/ Locate rel. extrema for $f(x) = x^3 - 3x^2$

Step 1 Find critical numbers

$$f'(x) = 3x^2 - 6x = 3x(x-2) \overset{\text{set}}{=} 0$$

Hence, $x = 0, 2$

Step 2 Compute $f''(c)$ for all crit. numbers

$$f''(x) = 6x - 6 = 6(x-1)$$

$$f''(0) = 6(-1) = -6 < 0$$

$$f''(2) = 6(1) = 6 > 0$$

Conclusion: f has a rel. max at $x=0$ and a rel. min. at $x=2$

Ex4/ Repeat for $f(x) = e^x(x-7)$

(1) Find crit. nums.

$$f'(x) = e^x(x-7) + e^x = e^x(x-7+1) = \boxed{e^x(x-6)} \overset{\text{set}}{=} 0$$

Either $e^x = 0$ OR $x-6=0$

Impossible $\boxed{x=6}$

(2) Find the sign of $f''(6)$

$$f''(x) = e^x(x-6) + e^x = e^x(x-5)$$

$$f''(6) = e^6(6-5) = e^6 > 0$$

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Hence, f has a rel. min. at $x=6$

Optionally Find the minimum value

$$f(6) = e^6(6-7) = -e^6$$

Ex5/ Repeat for $\frac{x}{x^2+4}$

$$f'(x) = \frac{x^2+4 - x(2x)}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2} \overset{\text{set}}{=} 0$$

$$\Rightarrow x = \pm 2$$

Compute the sign of $f''(-2)$ & $f''(2)$

$$f''(x) = \frac{(-2x)(x^2+4)^2 - (4-x^2)\,2(x^2+4)(2x)}{(x^2+4)^4}$$

$$= \underbrace{x\left[\frac{-2(x^2+4)^2}{(x^2+4)^4}\right]}_{\text{Always negative}} - \underbrace{(4-x^2)\frac{2(x^2+4)(2x)}{(x^2+4)^4}}_{0 \text{ at } x=\pm 2}$$

$f''(-2) > 0 \Rightarrow f$ has a rel. min at $x=-2$

$f''(2) < 0 \Rightarrow f$ has a rel. max at $x=2$