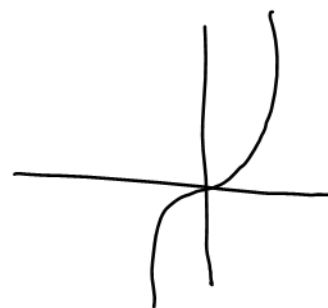
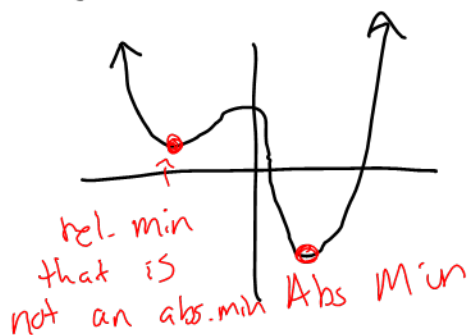
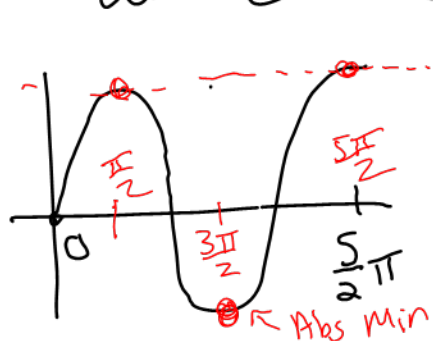


Lecture 19: The Extreme Value Theorem

Goal: Be able to compute the maximum and minimum of a differentiable function on a closed and bounded domain.

Definition: let f be a function with domain D :

- ① We say f has an absolute (or global) maximum at c if $f(x) \leq f(c)$ for all x in D .
- ② Likewise, f has an absolute minimum at c if $f(c) \leq f(x)$ for all x in D .



Theorem (Extreme Value Theorem) Let f be a function defined on a closed interval $[a, b]$.

- ① If f is continuous on $[a, b]$, then f has both an absolute maximum and minimum
- ② If f is differentiable on (a, b) , then the abs. max/min can occur in 3 places

(i) $x = a$

(ii) $x = b$

(iii) At a critical number of f on (a, b)



Ex 1 Find the max/min of $f(x) = x^3 - 3x^2 + 1$ on $[-\frac{1}{2}, 4]$

Step 1 Find critical numbers

$$f'(x) = 3x^2 - 6x = 3x(x-2) \stackrel{\text{set}}{=} 0$$

$$\Rightarrow x = 0, 2$$

Step 2 Compute the value of f at the endpoints and at each critical number

x	$-\frac{1}{2}$	0	2	4
$f(x)$	$\frac{1}{8}$	1	$\underbrace{-3}_{\text{min}}$	$\underbrace{17}_{\text{max}}$

$$f(x) = x^3 - 3x^2 + 1$$

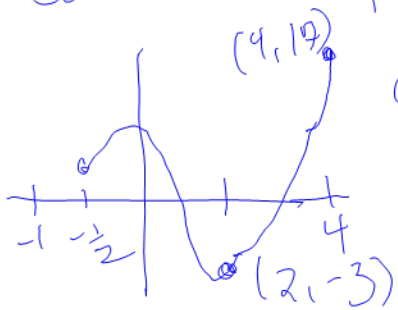
$$f(2) = 8 - 12 + 1 = -3$$

$$f(4) = 64 - 48 + 1 = 17$$

$$f(-\frac{1}{2}) = -\frac{1}{8} - \frac{3}{4} + 1 = \frac{1}{8}$$

$$\begin{array}{r} 64 \\ -48 \\ \hline 16 \end{array}$$

Conclusion: f has an absolute min at $(2, -3)$ and an absolute max at $(4, 17)$



Example 2 Repeat for $f(x) = x^4 - 2x^2 + 3$ on $[0, 2]$

Step 1 Find Critical Numbers

$$\begin{aligned} f'(x) &= 4x^3 - 4x = 4x(x^2 - 1) \\ &= 4x(x-1)(x+1) \stackrel{\text{set}}{=} 0 \end{aligned}$$

$$\Rightarrow x = -1, 0, 1$$

$x \in [0, 2]$ $\Rightarrow x = 0, 1$

Step 2

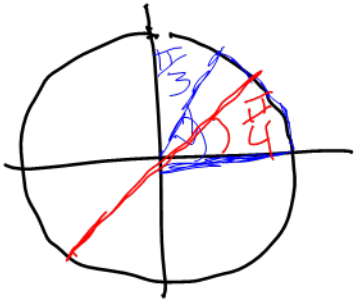
$$f(x) = x^4 - 2x^2 + 3$$

x	0	1	2	$f(1) = 1 - 2 + 3$
$f(x)$	3	<u>2</u> min	<u>11</u> max	$f(2) = 16 - 8 + 3$

Conclusion: On $[0, 2]$, f has an abs. min at $(1, 2)$ and a abs. max at $(2, 11)$

Example 3 Repeat for $f(x) = \sin x + \cos x$ on $[0, \frac{\pi}{3}]$

Step 1 $f'(x) = \cos x - \sin x \stackrel{\text{set}}{=} 0$
 $\sin x = \cos x$
 $\tan x = 1$
 $x = \pi/4$



Step 2

x	0	$\pi/4$	$\frac{\pi}{3}$
$f(x)$	1	$\sqrt{2}$ ≈ 1.414	$\frac{\sqrt{3}}{2} + \frac{1}{2}$ ≈ 1.366

$\sin(\frac{\pi}{4}) + \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$
 $\sin(\frac{\pi}{3}) + \cos(\frac{\pi}{3})$

Conclusion: On $[0, \frac{\pi}{3}]$, f has an abs min at $(0, 1)$ and an abs. max at $(\pi/4, \sqrt{2})$

Example 4 (HW19, Q8) Find the absolute extrema of $f(x) = 5x^2 e^x + 8$ on $[-4, 2]$

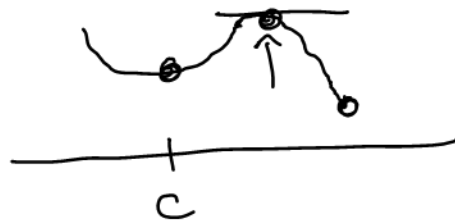
Step 1 $f'(x) = 10x e^x + 5x^2 e^x$
 $= 5x e^x (2 + x) \stackrel{\text{set}}{=} 0$

$$\Rightarrow x = 0, -2$$

Step 2

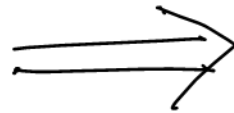
x	-4	-2	0	2
$f(x)$	$\frac{80}{e^4} + 8$ ≈ 9.465	$\frac{20}{e^2} + 8$ ≈ 16.707	8 min	$20e^2 + 8$ ≈ 155.78 Max

When f has Only 1 Critical Number



Point: Regardless of the interval, if f only has 1 critical number

Local
Extremum



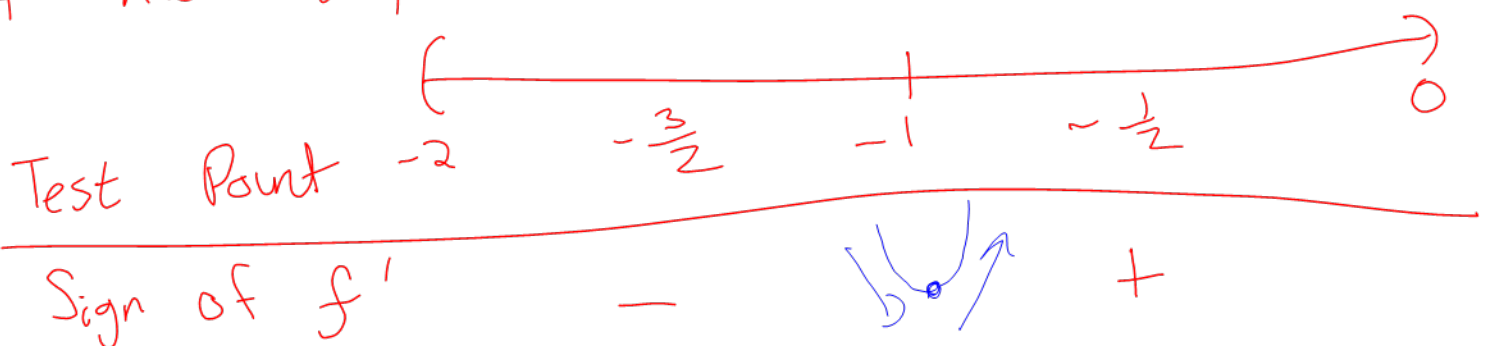
Absolute
Extremum

Example 5 Consider $f(x) = x^2 + 2x$ on $(-2, 0)$

$$f'(x) = 2x + 2 \stackrel{\text{set}}{=} 0$$

$$x = -1$$

f has only 1 crit. num. on $(-2, 0)$



$\Rightarrow$ Relative Minimum at -1 by $1^{st} \frac{d}{dx}$ test

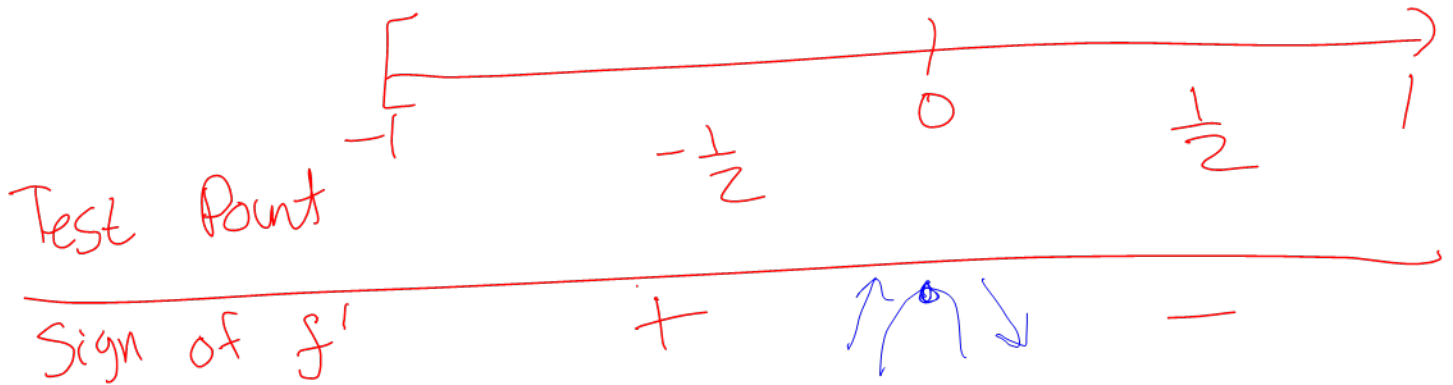
$\Rightarrow$ Abs. Min at -1

This says nothing about an abs. max.
We need to do extra work



Example 6 Consider $y = \frac{1}{4x^2+3}$ on $[-1, 1]$
 $= [4x^2+3]^{-1}$

$$y' = -\frac{1}{(4x^2+3)^2} (8x) \stackrel{\text{set}}{=} 0$$
$$-8x = 0$$
$$x = 0$$



$\Rightarrow$ Rel. max at $x = 0$

$\Rightarrow$ Abs. max at $x = 0$

