

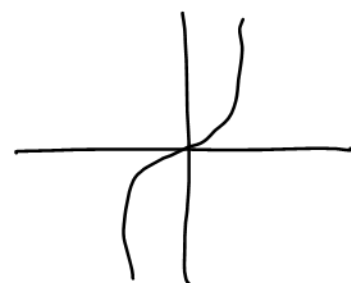
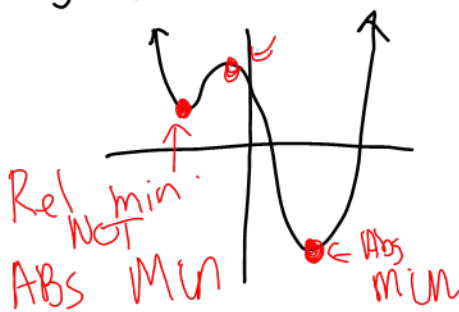
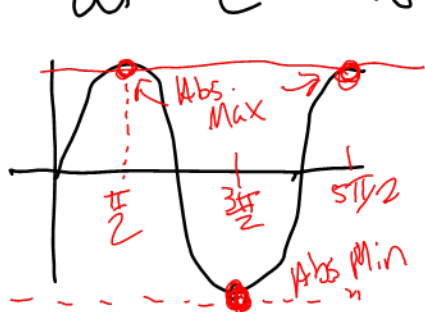
Lecture 19: The Extreme Value Theorem

Goal: Be able to compute the maximum and minimum of a differentiable function on a closed and bounded domain.

Definition: Let f be a function with domain D :

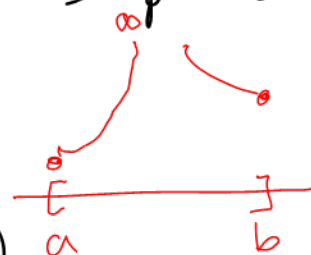
① We say f has an absolute (or global) maximum at c if $f(x) \leq f(c)$ for all x in D .

② Likewise, f has an absolute minimum at c if $f(c) \leq f(x)$ for all x in D .



Theorem (Extreme Value Theorem) Let f be a function defined on a closed interval $[a, b]$.

- ① If f is continuous on $[a, b]$, then f has both an abs. max and an abs. min.
- ② If f is differentiable on (a, b) , then the abs. max/min can only occur in 3 places:
- (i) $x = a$
 - (ii) $x = b$
 - (iii) At a critical number on (a, b)



Example 1 Find the maximum and minimum of $f(x) = x^3 - 3x^2 + 1$ on $[-\frac{1}{2}, 4]$

Step 1 Find our critical numbers

$$f'(x) = 3x^2 - 6x = 3x(x-2) \stackrel{\text{set}}{=} 0$$

$$\Rightarrow x = 0, 2$$

Step 2 Compute the value of f at the endpoints and at each critical number

x	$-\frac{1}{2}$	0	2	4
$f(x)$	$\frac{1}{8}$	1	$-3 \leftarrow \text{Min}$	$17 \leftarrow \text{Max}$

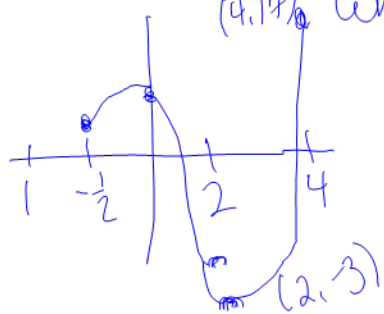
$$f(2) = 8 - 12 + 1 = -3$$

$$f(4) = 64 - 48 + 1 = 17$$

$$f(-\frac{1}{2}) = -\frac{1}{8} - \frac{3}{4} + 1 = \frac{1}{8}$$

$$\begin{array}{r} \sqrt{64} \\ -48 \\ \hline 16 \\ +1 \\ \hline 17 \end{array}$$

Conclusion: f has an absolute min at $(2, -3)$ and an abs. max at $(4, 17)$



Example 2 Repeat for

$$f(x) = x^4 - 2x^2 + 3 \text{ on } [0, 2]$$

Step 1 $f'(x) = 4x^3 - 4x = 4x(x^2 - 1)$

$$= 4x(x-1)(x+1) \stackrel{\text{set}}{=} 0$$

$$\Rightarrow x = -1, 0, 1$$

$$\underline{\underline{x \in [0, 2]}} \Rightarrow x = 0, 1$$

x	0	1	2
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$$x^4 - 2x^2 + 3 \quad f(x) \mid 3 \mid \quad \textcircled{2} \nearrow \text{min} \quad \textcircled{11} \nwarrow \text{max}$$

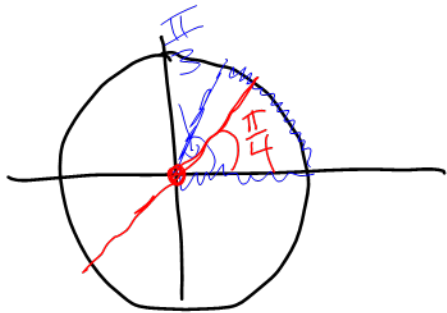
$$f(1) = 1 - 2 + 3$$

$$f(2) = 16 - 8 + 3$$

Conclusion : f has an abs. min at $(1, 2)$
and an abs. max at $(2, 11)$

Example 3 Repeat for $f(x) = \sin x + \cos x$
on $[0, \frac{\pi}{3}]$

Step 1 $f'(x) = \cos x - \sin x \stackrel{\text{set}}{=} 0$
 $\sin x = \cos x$



$$x \in [0, \frac{\pi}{3}] \quad \tan x = 1$$

$$\Rightarrow x = \frac{\pi}{4}$$

Step 2

x	0	$\pi/4$	$\pi/3$
$f(x)$	1 $\uparrow$ min	$\sqrt{2} \approx 1.414$ $\nwarrow$ max	$\frac{\sqrt{3}}{2} + \frac{1}{2} \approx 1.366$

$$f(x) = \sin x + \cos x$$

$$f(0) = 0 + 1$$

$$f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$f\left(\frac{\pi}{3}\right) = \sin \frac{\pi}{3} + \cos \frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{1}{2}$$

Conclusion: f has an abs. min at $(0, 1)$ and
an abs max at $(\frac{\pi}{4}, \sqrt{2})$

Ex 4 (HW19, Q8) Find the absolute extrema of $f(x) = 5x^2 e^x + 8$ on $[-4, 2]$

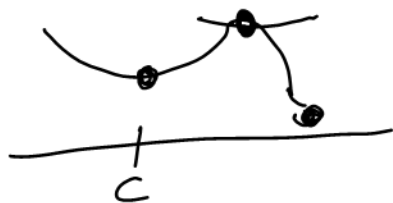
$$f'(x) = 10x e^x + 5x^2 e^x = 5x e^x (2+x) \stackrel{\text{set}}{=} 0$$

$$\Rightarrow x = -2, 0$$

Step 2

x	-4	-2	0	2
$f(x)$	$\frac{80}{e^4} + 8$ ≈ 9.465	$\frac{20}{e^2} + 8$ ≈ 16.707	8 ↑ Min	$20e^2 + 8$ ≈ 155.78 ↑ Max

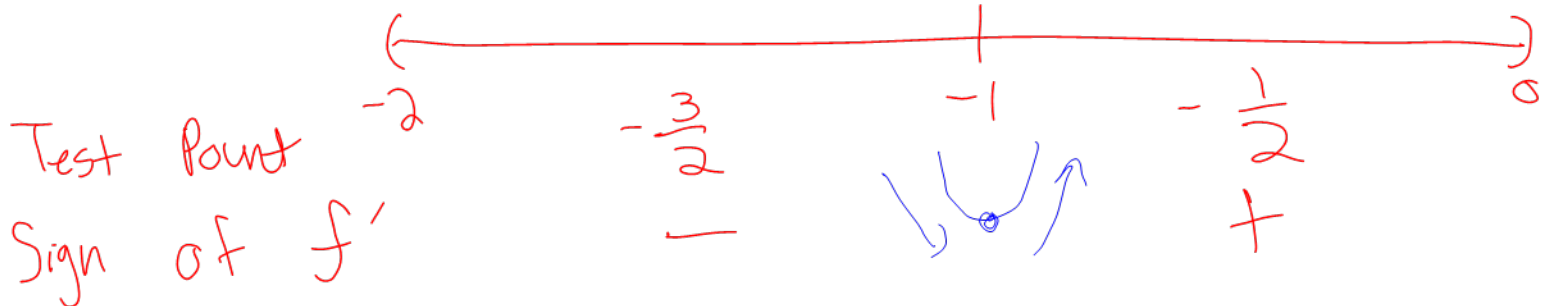
When f has Only 1 Critical Number



Point: Regardless of the interval, if f has only 1 critical number. Then

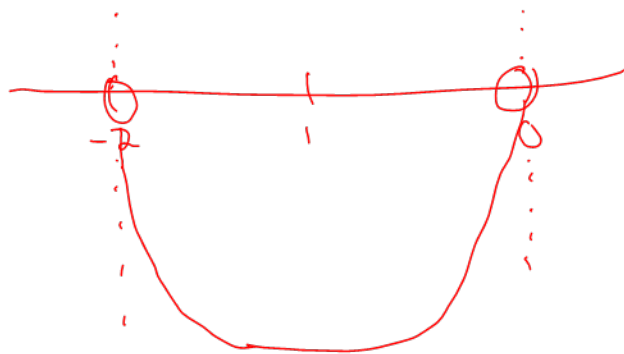
Relative Extremum $\Rightarrow$ Absolute Extremum

Ex 5 Consider $f(x) = x^2 + 2x$ on $(-2, 0)$
 $f'(x) = 2x + 2 \stackrel{\text{set}}{=} 0 \Rightarrow x = -1$



$\Rightarrow f$ has a rel. min at $x = -1$

$\Rightarrow f$ has an abs. min at $x = -1$



Ex 6 Consider $f(x) = \frac{1}{4x^2+3}$ on $[-1, 1)$

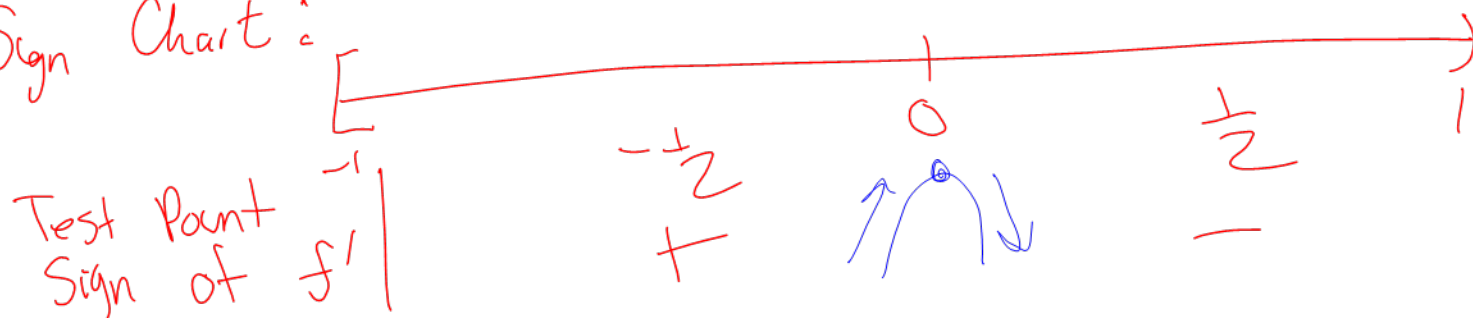
$= [4x^2+3]^{-1}$

$$f'(x) = \frac{-1}{[4x^2+3]^2} \cdot (8x) \quad \underline{\text{set } 0}$$

$$-8x = 0$$

$$x = 0$$

Sign Chart:



$\Rightarrow f$ has a rel. max at $x = 0$

$\Rightarrow f$ has a abs. max at $x=0$

