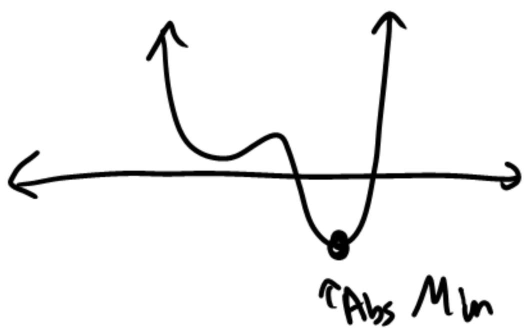
Lecture 19: The Extreme Value Theorem

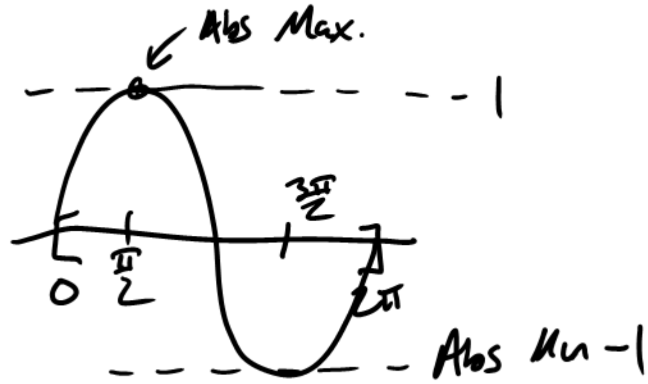
Goal: Be able to compute the maximum and minimum of a differentiable function on a closed and bounded domain.

Def Let f be a fcn defined on a domain D

① f has an absolute (or global) maximum at c if $f(c) \geq f(x)$ for x in D.

② f has an absolute min when $f(c) \leq f(x)$ for x in D.





Theorem (The Extreme Value Theorem) Let f be defined on a closed interval $I = [a,b]$

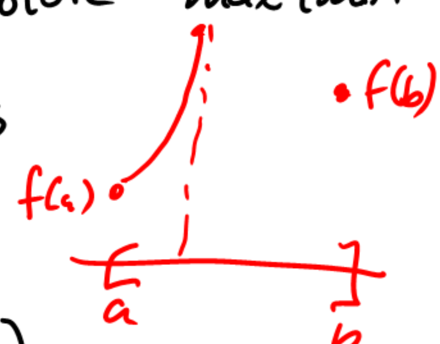
① If f is continuous, then an absolute max/min exists.

② If f is differentiable on (a,b), the absolute max/min can only occur at 3 places

(i) $x=a$

(ii) $x=b$

(iii) A critical number on (a,b)



Ex/ Find the maximum and minimum value of $\boxed{f(x) = x^3 - 3x^2 + 1}$ on $[-\frac{1}{2}, 4]$

① Locate any critical numbers

$$f'(x) = 3x^2 - 6x = 3x(x-2) \overset{\text{set}}{=} 0 \Rightarrow x = 0, 2$$

② Compare the y values at the crit. numbers and endpoints

x	0	2 (Min)	$-\frac{1}{2}$	4 (Max)
$f(x) = x^3 - 3x^2 + 1$	1	-3	$\frac{1}{8}$	17

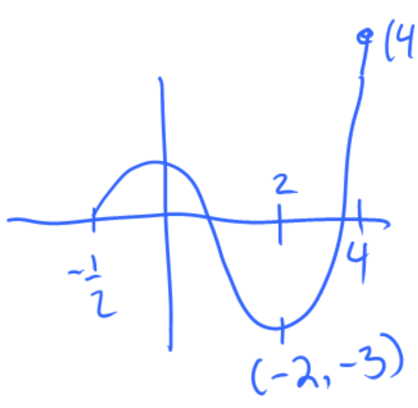
$f(2) = 8 - 12 + 1 = -3$

$f(-\frac{1}{2}) = -\frac{1}{8} - 3(\frac{1}{4}) + 1 = \frac{1}{8}$

$f(4) = 64 - 3 \cdot 16 + 1$

$$\begin{array}{r} 64 \\ -48 \\ \hline 16 \\ +1 \\ \hline 17 \end{array}$$

Conclusion: f has a minimum value of -3 at $x = 2$ and a max value of 17 at $x = 4$



Ex/ Repeat for $f(x) = x^4 - 2x^2 + 3$ on $[0, 2]$

① Critical Numbers

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1)$$

$$= 4x(x+1)(x-1) \overset{\text{set}}{=} 0 \Rightarrow x = -1, 0, 1$$

Outside $[0,2]$

Only care about $x = 0, 1$

② Make table

Critical Num

x	0	1	2
$f(x)$	3	2	11

Endpoints

Min Max

<u>Conclusion:</u> f has a min value of 2 at $x=1$, while f has a max value of 11 at $x=2$

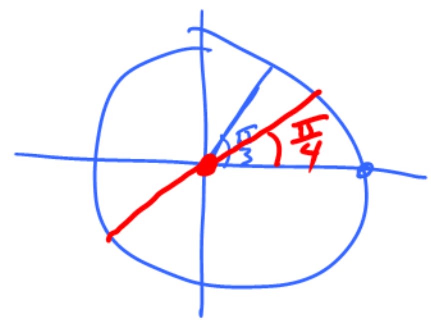
Ex 3/ Repeat for $f(x) = \sin x + \cos x$ on $[0, \frac{\pi}{3}]$

① Crit. Nums

$$f'(x) = \cos x - \sin x \overset{\text{set}}{=} 0$$

$$\cos x = \sin x$$

$$\tan x = 1$$



$\Rightarrow x = \frac{\pi}{4}$

② Make Table

x	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$f(x)$	1	$\sqrt{2} \approx 1.414$	$\frac{\sqrt{3}}{2} + \frac{1}{2}$
	1	Max	≈ 1.366

$f(x) = \sin x + \cos x$

<u>Conclusion:</u> f obtains a max of $\sqrt{2}$ at $x = \frac{\pi}{4}$ while f obtains a min of 1 at $x = 0$

<u>When a fcn only has one critical value</u>

Point: If a fcn has only one critical value, relative extremum $\Longrightarrow$ absolute extremum

Ex 4/ Consider $\frac{\ln x}{x}$ on $(0, \infty)$

$$\left[\frac{\ln x}{x}\right]' = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1-\ln x}{x^2} \stackrel{\text{set}}{=} 0 \Rightarrow \ln x = 1 \Rightarrow x = e$$

In other words, f has a local maximum at $x=e$

What we can say: f has an absolute maximum at $x=e$

We can't say anything at a minimum w/o further investigative work

Ex 5/ $y = x^2 + 2x$ on $(-2, 0)$

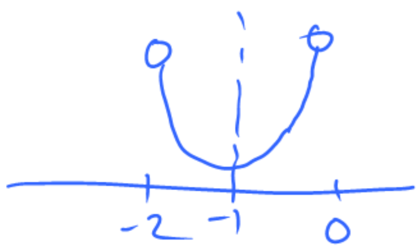
$y' = 2x + 2 \stackrel{\text{set}}{=} 0 \Longrightarrow \boxed{x = -1}$

$x=-1$ is the loc of a local min

Thus, f has an absolute min at $x=-1$

[provided the domain is $(-2,0)$]

We can't say anything about a max



Ex 6/ $y = \dfrac{1}{(4x^2+3)}$ on the interval $[-1,1)$

$$y' = -\frac{1}{(4x^2+3)^2}(8x) \overset{set}{=} 0 \implies \boxed{x=0}$$

$f'(-\tfrac{1}{2}) > 0$ $\quad$ $f'(\tfrac{1}{2}) < 0$

$[-1 \quad -\tfrac{1}{2} \quad 0 \quad \tfrac{1}{2} \quad 1)$

By the 1^{st} derivative test, y has a local max at $x=0$. Thus, an absolute max.

Without further work, we can't say if there is a min