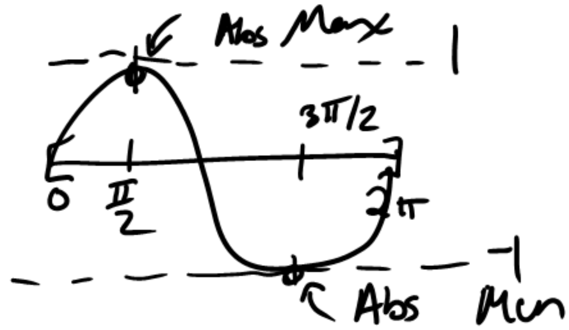
Goal: Be able to compute the maximum and minimum of a differentiable function on a closed and bounded domain.

__Def__ Let f be a function on a domain D

(1) f has an __absolute__ (or __global__) maximum at c if $f(c) \geq f(x)$ for all x in D

(2) f has an absolute min at c if $f(c) \leq f(x)$ for all x in D



__Thm (The Extreme Value Theorem)__ Let f be a fcn defined on a closed interval $I = [a,b]$.

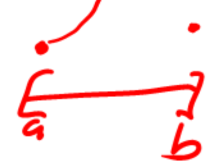
(1) If f is continuous on $[a,b]$, then there is at least one absolute max and min.

(2) If f is differentiable on (a,b), then there are only 3 places the abs. max/min can occur

(i) $x=a$

(ii) $x=b$

(iii) At a critical number on (a,b).



Ex1/ Determine the location of the maximum and minimum value of $f(x) = x^3 - 3x^2 + 1$ on $[-\frac{1}{2}, 4]$

① Find Critical Numbers

$$f'(x) = 3x^2 - 6x = 3x(x-2) \overset{\text{set}}{=} 0 \Rightarrow x = 0, 2$$

② Make a Table

x	0	2	$-\frac{1}{2}$	4
$f(x)$	1	-3	$\frac{1}{8}$	17
		Min		Max

$f(2) = 8 - 12 + 1 = -3$

$f(-\frac{1}{2}) = -\frac{1}{8} - \frac{3}{4} + 1 = \frac{1}{8}$

$f(4) = 64 - 48 + 1$

Conclusion: f has a minimum value of -3 at $x=2$ and f has a max value of 17 at $x=4$

Ex2/ Find the max/min of the $f(x) = x^4 - 2x^2 + 3$ on $[0,2]$

① Crit. Values on $[0,2]$

$$f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1) \overset{\text{set}}{=} 0$$

$\Rightarrow x = -1, 0, 1$ ← Not in $[0,2]$

$\boxed{x = 0, 1}$

② Make Table

x	0	1	2
$f(x)$	3	2	11
		Min	Max

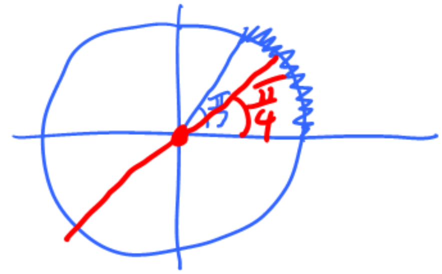
Conclusion: f has a minimum value of 2 at $x=1$ while f has a max value of 11 at $x=2$

Ex 3/ Repeat for $f(x) = \sin x + \cos x$ on $[0, \frac{\pi}{3}]$

$$f'(x) = \cos x - \sin x \overset{\text{set}}{=} 0$$

$$\cos x = \sin x$$

$$\tan x = 1$$



$\Rightarrow x = \frac{\pi}{4}$ $\qquad f(x) = \sin x + \cos x$

② Make Table

x	$f(x)$
$\pi/4$	$\sqrt{2} \approx 1.414$ ← Max
0	1 ← Min
$\pi/3$	$\frac{\sqrt{3}}{2} + \frac{1}{2} \approx 1.366$

Conclusion: On $[0, \frac{\pi}{3}]$, f has a maximum value of $\sqrt{2}$ at $x = \frac{\pi}{4}$ while f has a min of 1 at $x = 0$

When f has only one critical value

Point: If f has only one critical value

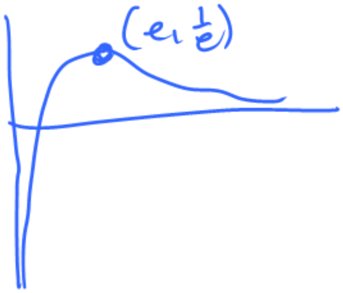
Relative Extremum $\Longrightarrow$ Absolute Extremum

Ex 4/ Consider $f(x) = \frac{\ln x}{x}$ on $(0, \infty)$

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} \overset{\text{set}}{=} 0 \Rightarrow \ln x = 1 \Rightarrow x = e$$

By the 1st Derivative Test, f has a local max at $x = e$.

Thus, $(e, \frac{1}{e})$ is an absolute max of f



We can't say anything about a minimum without further work.

Ex 5/ $f(x) = x^2 + 2x$ on $(-2, 0)$

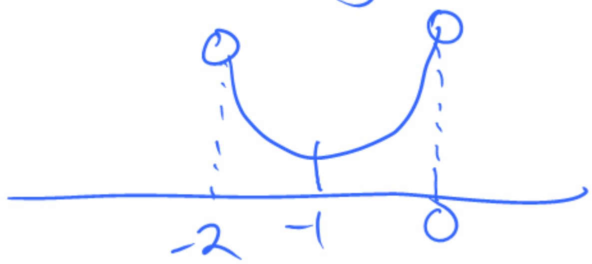
① Find the lone crit. num.

$$f'(x) = 2x + 2 = 2(x+1) \overset{\text{set}}{=} 0 \Rightarrow \boxed{x = -1}$$

f has a local minimum at $x = -1$

$\Rightarrow$ f has an absolute min at $x = -1$

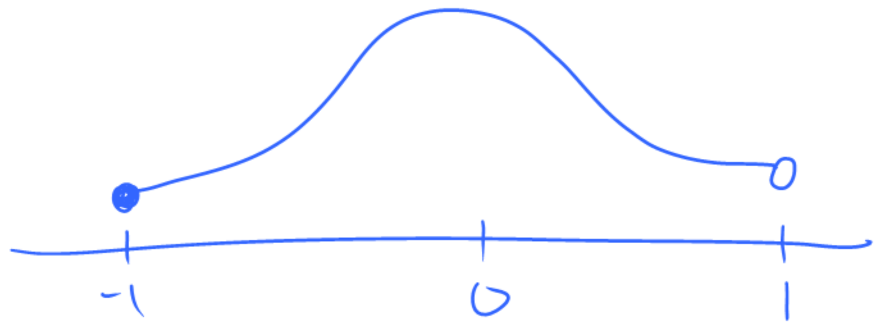
The work says nothing about a max.



Ex 6/ $y = \dfrac{1}{4x^2 + 3}$ on $[-1, 1)$

$$y' = -\frac{1}{(4x^2+3)^2}(8x) \overset{\text{set}}{=} 0 \Rightarrow -8x = 0 \Rightarrow x = 0$$

y has a local (hence absolute) max at 0 by the 1^{st} Derivative Test


-1
0
1