

Lecture 21: Limits at Infinity

Goal: Understand the end behavior (asymptotic behavior) of functions.

Definition. We say:

$$\lim_{x \rightarrow \infty} f(x) = L$$

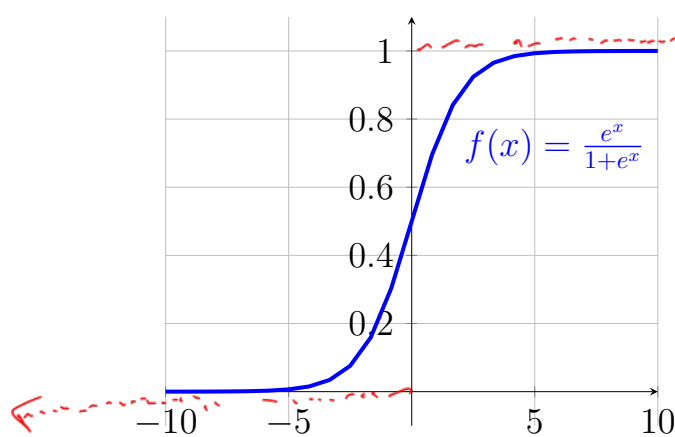
to mean $f(x)$ goes to L as x “goes to infinity” (i.e., x gets very, very large). Similarly,

$$\lim_{x \rightarrow -\infty} f(x) = L$$

means $f(x)$ goes to L as x “goes to negative infinity” (i.e., x gets very, very “largely negative”).

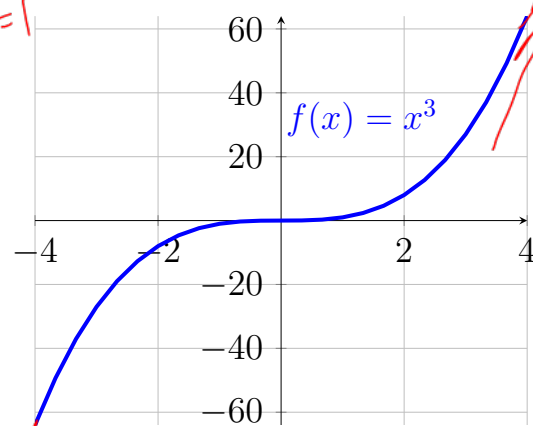
NOTE: A number being “small” typically refers to when it is close to 0. Hence, why I am using the phrase “largely negative,” even though it is a bit awkward.

Example 1. For each part, determine $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$, where the graph of f is given.



$$\lim_{x \rightarrow \infty} f(x) = 1$$

$$\lim_{x \rightarrow -\infty} f(x) = 0$$



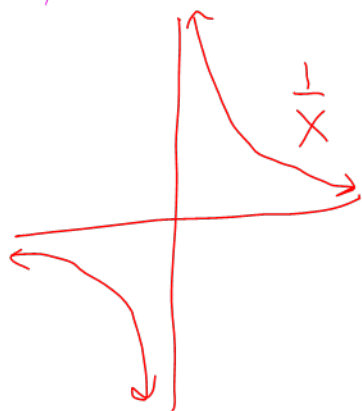
$$\lim_{x \rightarrow \infty} f(x) = \infty$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

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Example 2. Determine the end behavior (limits at infinity) for the function

$$f(x) = \frac{1}{x}$$



x	$f(x)$
-1	$-\frac{1}{1} = -1$
-100	$-\frac{1}{100} = -0.01$
-10000	$-\frac{1}{10000} = -0.0001$
-1000000	$-\frac{1}{1000000} = -0.000001$

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

In general, for any positive integer n ,

$$\frac{1}{x^2}, \frac{1}{x^7}$$

$$\lim_{x \rightarrow \infty} \frac{1}{x^n} = \lim_{x \rightarrow -\infty} \frac{1}{x^n} = 0$$

$$\frac{1}{x^n}$$

Example 3. Compute $\lim_{x \rightarrow \infty} \left(2 + \frac{3}{x^2} \right)$

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(2 + \frac{3}{x^2} \right) &= \lim_{x \rightarrow \infty} 2 + \lim_{x \rightarrow \infty} \frac{3}{x^2} \\ &= \lim_{x \rightarrow \infty} 2 + 3 \underbrace{\lim_{x \rightarrow \infty} \frac{1}{x^2}}_{\rightarrow 0} \\ &= 2 + 0 \\ &= 2 \end{aligned}$$

Rational Functions

Main Strategy (Formally):

Factor out the largest power of x in the denominator



Example 4. For $f(x) = \frac{x^2 - 1}{2x^2 + 1}$, compute $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$.

$$\lim_{x \rightarrow \infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \rightarrow \infty} \frac{x^2 \left(\frac{x^2}{x^2} - \frac{1}{x^2} \right)}{x^2 \left(\frac{2x^2}{x^2} + \frac{1}{x^2} \right)} = \lim_{x \rightarrow \infty} \frac{1 - \frac{1}{x^2}}{2 + \frac{1}{x^2}} = \frac{1 - 0}{2 + 0} = \frac{1}{2}$$

$$\lim_{x \rightarrow -\infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \rightarrow -\infty} \frac{1 - \frac{1}{x^2}}{2 + \frac{1}{x^2}} = \frac{1}{2}$$

Example 5. For $f(x) = \frac{x^2 + 2}{x^3 + x^2 - 1}$, compute $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$.

$$\lim_{x \rightarrow \infty} \frac{x^2 + 2}{x^3 + x^2 - 1} = \lim_{x \rightarrow \infty} \frac{x^3 \left(\frac{x^2}{x^3} + \frac{2}{x^3} \right)}{x^3 \left(\frac{x^3}{x^3} + \frac{x^2}{x^3} - \frac{1}{x^3} \right)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{2}{x^3}}{1 + \frac{1}{x} - \frac{1}{x^3}} = \frac{0}{1} = 0$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x} + \frac{2}{x^3}}{1 + \frac{1}{x} - \frac{1}{x^3}} = \frac{0}{1} = 0$$

Main Strategy (Informally):

Compare the ratio of the leading terms

Example 6. For $f(x) = \frac{x^2 + x}{-x + 3}$, compute $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$.

$$\lim_{x \rightarrow \infty} \frac{x^2 + x}{-x + 3} = \lim_{x \rightarrow \infty} \frac{x^2}{-x} = \lim_{x \rightarrow \infty} -x = -\infty$$

$$\lim_{x \rightarrow -\infty} \frac{x^2 + x}{-x + 3} = \lim_{x \rightarrow -\infty} \frac{x^2}{-x} = \lim_{x \rightarrow -\infty} -x = \infty$$

In Summary: For a rational function:

$$f(x) = \frac{a(x)}{b(x)}$$

if $LT(a(x))$ and $LC(a(x))$ denote the leading term and leading coefficient of the polynomial $a(x)$ respectively, then,

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{LT(a(x))}{LT(b(x))}$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{LT(a(x))}{LT(b(x))}$$

$$3x^2 + 1 = a(x)$$
$$LT(a(x)) = 3x^2$$
$$LC(a(x)) = 3$$

In particular,

1. If $\deg(a(x)) = \deg(b(x))$, then:

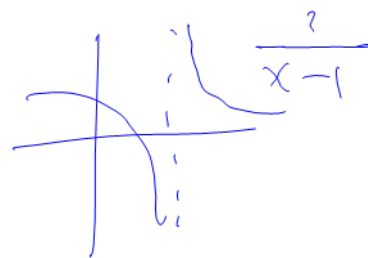
$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = \frac{LC(a(x))}{LC(b(x))}$$

2. If $\deg(a(x)) < \deg(b(x))$, then:

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = 0$$

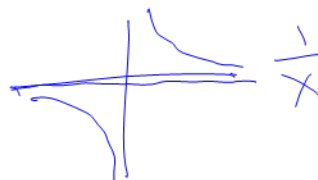
3. If $\deg(a(x)) > \deg(b(x))$, more work is needed (since it depends on whether $\deg(a(x)) - \deg(b(x))$ is even or odd as well as whether $\frac{LC(a(x))}{LC(b(x))}$ is positive or negative).

Horizontal and Vertical Asymptotes (HAs and VAs)



Definition. A function $f(x)$ has a **vertical asymptote** at $x = c$ if either $\lim_{x \rightarrow c^-} f(x)$ or $\lim_{x \rightarrow c^+} f(x)$ is NOT finite.

Definition. A function $f(x)$ has a **horizontal asymptote** at $y = L$ if either $\lim_{x \rightarrow -\infty} f(x) = L$ or $\lim_{x \rightarrow \infty} f(x) = L$.



Example 7. Find the locations of any Horizontal and Vertical Asymptotes

for $f(x) = \frac{3x^2 + 5}{x^2 - 4}$

For a rational function, VAs occur when the denominator is 0, but the numerator is non-zero. [After removing common factors]

$$x^2 - 4 \neq 0$$

$$x^2 = 4$$

$$\Rightarrow x = \pm 2$$

Numerator is not 0 at -2 and 2 $\Rightarrow$ VAs at $x = -2, 2$

For a rational function $f(x)$, HAs occur when either $\lim_{x \rightarrow -\infty} f(x)$ or $\lim_{x \rightarrow \infty} f(x)$ is finite.

NOTE: Rational functions have at most one HA.

$$\lim_{x \rightarrow \infty} \frac{3x^2 + 5}{x^2 - 4} = \lim_{x \rightarrow \infty} \frac{3x^2}{x^2} = \lim_{x \rightarrow \infty} 3 = 3$$

$y = 3$ is a HA

$$\lim_{x \rightarrow -\infty} f(x) = 3$$

Slant (Oblique) Asymptotes

Recall Polynomial Long Division:

Theorem (Polynomial Division Algorithm). For a rational function $\frac{a(x)}{b(x)}$ there are unique polynomials $q(x)$ (called the **quotient**) and $r(x)$ (called the **remainder**) such that:

$$\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)} \quad \text{AND} \quad \deg(r(x)) < \deg(b(x))$$

Compare this with numbers:

$$\begin{array}{r} 3 \\ 2 \overline{)7} \\ \underline{-6} \\ 1 \end{array}$$

$$7 = 3(2) + 1$$

$$\frac{7}{2} = 3 + \frac{1}{2}$$

Example 8. Find the quotient and remainder when $2x^2 + 3x + 1$ is divided by $x - 1$.

$$\begin{array}{r} 2x + 5 \\ x-1 \overline{)2x^2 + 3x + 1} \\ \underline{-(2x^2 - 2x)} \\ 5x + 1 \\ \underline{-(5x - 5)} \\ 6 \end{array}$$

$$\begin{array}{l} \deg(6) = 0 < 1 = \deg(x-1) \\ 2x^2 + 3x + 1 = \\ \underline{x-1} \\ (2x + 5) + \frac{6}{x-1} \end{array}$$

Definition. The line $y = mx + b$ is a **slant (oblique) asymptote** of the graph of $f(x)$ if:

$$\lim_{x \rightarrow \infty} [f(x) - (mx + b)] = 0 \quad \text{OR} \quad \lim_{x \rightarrow -\infty} [f(x) - (mx + b)] = 0$$

Informally, this means the graph of $f(x)$ gets closer and closer to the line $y = mx + b$ as x goes to positive or negative infinity.

Theorem. The following are equivalent for a rational function $\frac{a(x)}{b(x)}$:

1. $\frac{a(x)}{b(x)}$ has a slant asymptote.
2. The quotient in polynomial long division is a degree 1 polynomial.

$$3. \deg(a(x)) = \deg(b(x)) + 1$$

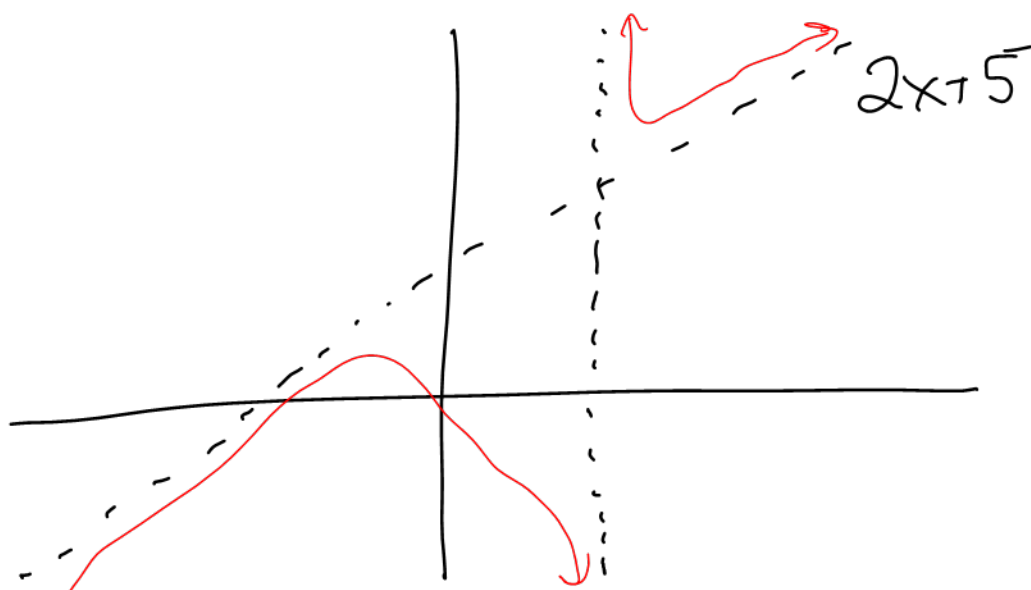
If these hold, the quotient obtained from long division is a slant asymptote.

Example 9. Does the function $f(x) = \frac{2x^2 + 3x + 1}{x - 1}$ have a slant asymptote? If so, find it.

Yes! Since the $\deg(\text{top}) = 2 = 1 + 1 = \deg(\text{bottom}) + 1$

$$\frac{2x^2 + 3x + 1}{x - 1} = (2x + 5) + \frac{6}{x - 1}$$

$\rightarrow 0 \text{ as } x \rightarrow \pm \infty$



Example 10. Find all asymptotes (HAs, VAs, and slant) for the function

$$f(x) = \frac{2x^3 - 3x + 1}{x^2 - x - 6}$$

No HAs since there is a slant asy.

Slant Asymptote:

$$\begin{array}{r}
 \text{Slant Asymptote: } \quad \quad \quad \boxed{2x - 2} \leftarrow \text{Slant Asymptote} \\
 x^2 - x - 6 \overline{) 2x^3 - 3x + 1} \\
 \underline{-(2x^3 - 2x^2 - 12x)} \\
 -2x^2 + 9x + 1 \\
 \underline{-(-2x^2 + 2x - 12)} \\
 7x + 13
 \end{array}$$

EDITOR'S NOTE: The polynomial division was indeed incorrect. See the HTML Notes for the correct version

VAs: $x^2 - x - 6 \stackrel{!}{=} 0 \Rightarrow x = -2, 3$
 $(x + 2)(x - 3)$

