Goal: Understand the end behavior (asymptotic behavior) of functions.

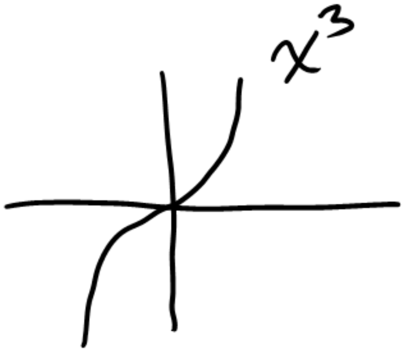
Def We say

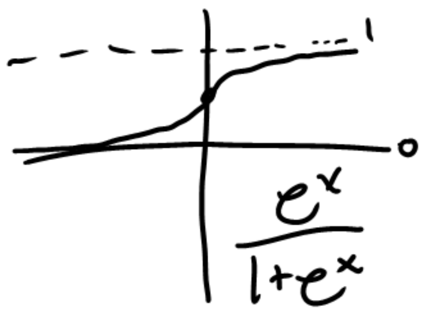
$$\lim_{x\to\infty} f(x) = L$$

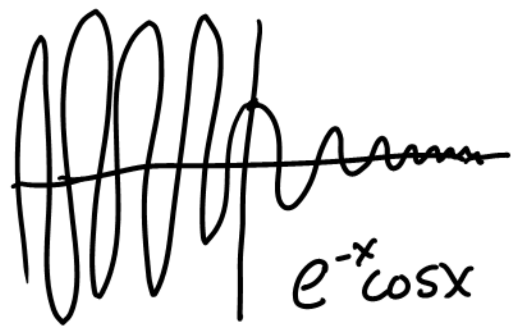
to mean $f(x) \to L$ as $x \to \infty$ (x gets really large)

Similiarly,

$$\lim_{x\to-\infty} f(x) = L$$







$\lim_{x\to\infty} f(x) = \infty$

$\lim_{x\to-\infty} f(x) = -\infty$

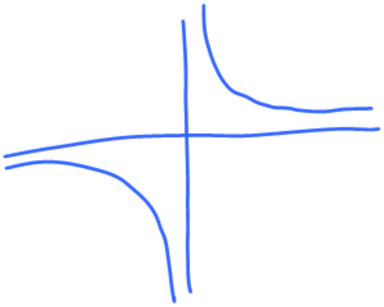
$\lim_{x\to\infty} f(x) = 1$

$\lim_{x\to-\infty} f(x) = 0$

$\lim_{x\to\infty} f(x) = 0$

$\lim_{x\to-\infty} f(x)$ DNE

Ex 1 $f(x) = \frac{1}{x}$



x	$f(x)$
1	1
100	$\frac{1}{100} = 0.01$
10 000	$\frac{1}{10000} = 0.0001$
1000 000	$\frac{1}{1000000} = 0.000001$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Similiarly,

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

In general

$$\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

for any positive integer n

Ex/ Find $\lim_{x\to\infty}\left(2+\frac{3}{x}\right)$

$$\lim_{x\to\infty}\left(2+\frac{3}{x}\right)=\lim_{x\to\infty}2+3\cdot\lim_{x\to\infty}\frac{1}{x}=2+3\cdot 0=2$$

Rational Functions

Main Strategy: Factor out the highest power of x in the denominator

Ex3/ For $f(x)=\frac{x^2-1}{x^2+1}$, find $\lim_{x\to\infty}f(x)$ and $\lim_{x\to-\infty}f(x)$

$$f(x)=\frac{x^2-1}{x^2+1}\ \overset{\text{when } x\neq 0}{=\!=\!=}\ \frac{\cancel{x^2}\left(\frac{x^2}{x^2}-\frac{1}{x^2}\right)}{\cancel{x^2}\left(\frac{x^2}{x^2}+\frac{1}{x^2}\right)}=\frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}$$

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{1-\cancelto{0}{\frac{1}{x^2}}}{1+\cancelto{0}{\frac{1}{x^2}}}=\frac{1}{1}=1$$

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}=\frac{1}{1}=1$$

Ex 4/ Repeat for $f(x)=\frac{x^2+x}{-x+3}$

$$f(x)=\frac{x^2+x}{-x+3}\ \underset{\text{+ when } x \text{ large}}{\overset{\text{when } x\neq 0}{=\!=\!=}}\ \frac{x(x+1)}{x\left(-1+\frac{3}{x}\right)}=\frac{x+1}{-1+\frac{3}{x}}$$

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{\overset{\infty}{x+1}}{\underbrace{-1+\frac{3}{x}}_{\text{Negative when } x \text{ is large}}}=-\infty$$

$\frac{3}{x}\to 0$, -1

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\overbrace{(x+1)}^{-\infty}}{\underbrace{\left(-1+\frac{3}{x}\right)}_{-1}} = \infty$$

Ex5/ Repeat for $f(x) = \dfrac{x^2+2}{x^3+x^2-1}$

$$x \text{ large } \frac{x^2}{x^3} = \frac{1}{x} \to 0$$

$$\underset{x\neq 0}{f(x) =} \frac{x^3\left(\frac{x^2}{x^3}+\frac{2}{x^3}\right)}{x^3\left(\frac{x^3}{x^3}+\frac{x^2}{x^3}-\frac{1}{x^3}\right)} = \frac{\frac{1}{x}+\frac{2}{x^3}}{1+\frac{1}{x}-\frac{1}{x^3}}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\underbrace{\frac{1}{x}}_{\to 0}+\underbrace{\frac{2}{x^3}}_{\to 0}}{1+\underbrace{\frac{1}{x}}_{\to 0}-\underbrace{\frac{1}{x^3}}_{\to 0}} = \frac{0+0}{1+0-0} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\frac{1}{x}+2/x^3}{1+1/x-1/x^3} = \frac{0+0}{1+0-0} = 0$$

Horizontal and Vertical Asymptotes (HAs and VAs)

Ex6/ Compute the HAs and VAs of $f(x) = \dfrac{3x^2+5}{x^2-4}$

VAs occur when the denominator is 0, but the numerator is non-zero

$$\text{Denominator} = x^2-4 \overset{\text{set}}{=} 0$$

$$(x-2)(x+2) = 0$$

$$\Rightarrow x = -2, 2 \leftarrow \text{Locations of VAs}$$

HAs occur when $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ are finite

$$\lim_{x\to\infty} \frac{3x^2+5}{x^2-4} = \lim_{x\to\infty} \frac{x^2\left(3+\frac{5}{x^2}\right)}{x^2\left(1-\frac{4}{x^2}\right)} = \lim_{x\to\infty} \frac{3+\frac{5}{x^2}}{1-\frac{4}{x^2}} = \frac{3}{1} = 3$$

NOTE: For rational fcns, there can only be 1 HA

$y = 3$ is the location of the HA

Slant (Oblique) Asymptotes Recall polynomial long division

For a rational function $\frac{a(x)}{b(x)}$, there are polynomials $q(x)$ and $r(x)$ where

$$\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)} \quad ; \text{ degree of } r < \text{ degree of } b$$

$$\begin{array}{r} 2 \\ 2\overline{)5} \\ -4 \\ \hline 1 \end{array}$$

$\frac{5}{2} = 2 + \frac{1}{2}$

Ex/

$$\begin{array}{r} 2x + 5 \\ x-1\overline{)2x^2 + 3x + 1} \\ -(2x^2 - 2x) \\ \hline 5x + 1 \\ -(5x - 5) \\ \hline 6 \end{array}$$

$\deg(6) < \deg(x-1)$

STOP

$$\frac{2x^2+3x+1}{x-1} = (2x+5) + \frac{6}{x-1}$$

Theorem The Following are equivalent

① A rational fcn $\frac{a(x)}{b(x)}$ has a slant asymptote

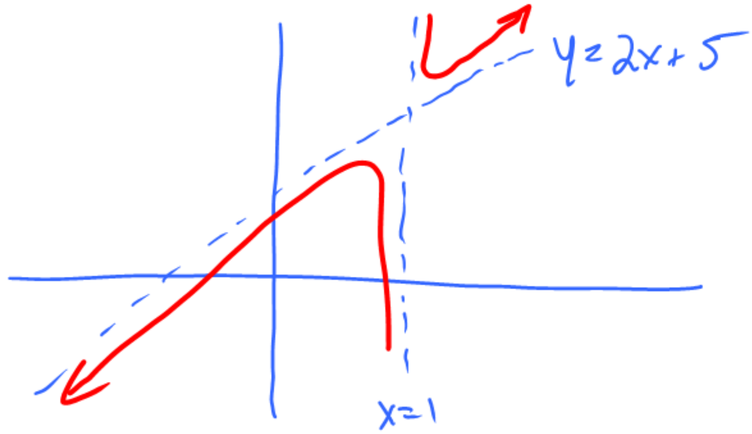
② The quotient in long division returns a deg 1 polynomial

③ $\deg(a(x)) = \deg(b(x)) + 1$

Ex 7/ $f(x) = \frac{2x^2 + 3x + 1}{x-1} = (2x+5) + \frac{6}{x-1}$

≈ 0 when x is large

$y = 2x+5$ is our asymptote



Ex8 Find all asymptotes of $f(x) = \dfrac{2x^3 - 3x + 1}{x^2 - x - 6}$ (if they exists)

$= (2x+2) + \dfrac{11x+13}{x^2-x-6}$

VAs: $x^2 - x - 6 \overset{\text{set}}{=} 0 \implies x = -2, 3$

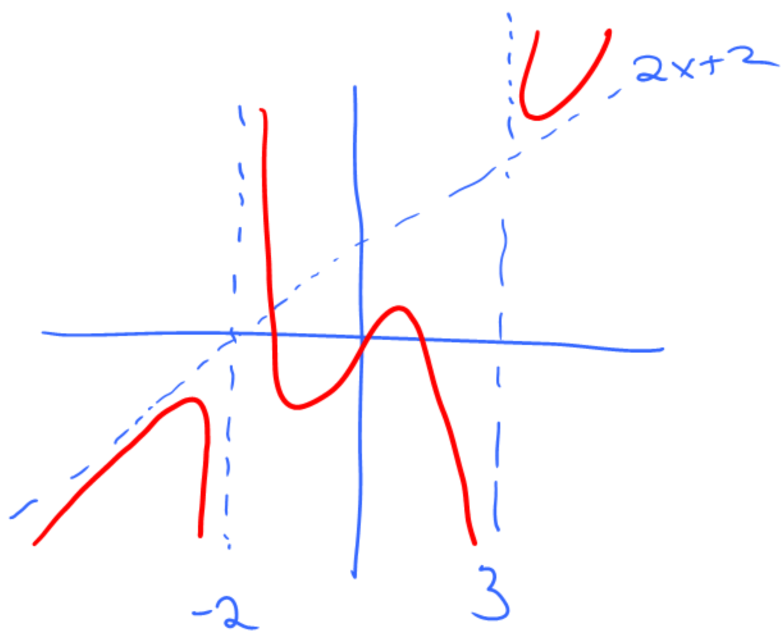
HAs: None

$y = 2x+2$ is slant asymptote

Slant:

$$
\begin{array}{r}
2x + 2 \\
x^2-x-6 \,\big)\, \overline{\cancel{2x^3} - 3x + 1} \\
-(\cancel{2x^3} - 2x^2 - 12x) \\
\hline
2x^2 + 9x + 1 \\
-(2x^2 - 2x - 12) \\
\hline
11x + 13
\end{array}
$$

STOP



Since $\deg(11x+13) = 1 < 2 = \deg(x^2-x-6)$