Goal: Understand the end behavior (asymptotic behavior) of functions.

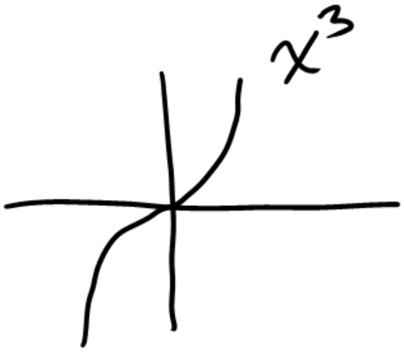
Def/ We say

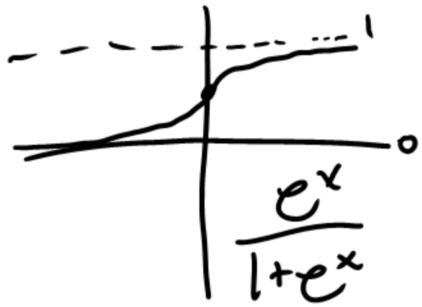
$$\lim_{x \to \infty} f(x) = L$$

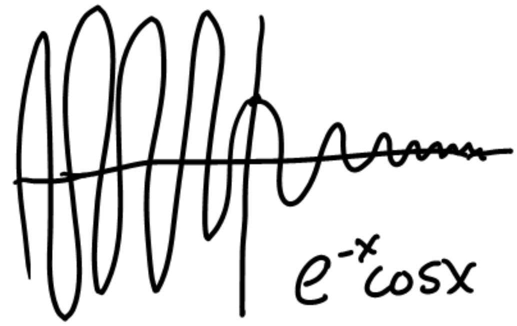
to mean $f(x) \to L$ as $x \to \infty$. (x gets really large)

Similiarly,

$$\lim_{x \to -\infty} f(x) = L$$







$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = -\infty$

$\lim_{x \to \infty} f(x) = 1$

$\lim_{x \to -\infty} f(x) = 0$

$\lim_{x \to \infty} f(x) = 0$

$\lim_{x \to -\infty} f(x)$ DNE

Ex1/ $f(x) = \frac{1}{x}$

x	$f(x)$
1	1
100	$1/100 = 0.01$
10000	$1/1000 = 0.0001$
1000 000	$\frac{1}{1000000} = 0.00001$

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Similarly,

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

Ex2/ Find $\lim_{x \to \infty} \left(2 + \frac{3}{x}\right)$

$$\lim_{x \to \infty} \left(2 + \frac{3}{x}\right) = \lim_{x \to \infty} 2 + 3 \cdot \lim_{x \to \infty} \frac{1}{x} = 2 + 3 \cdot 0 = 2$$

In general,

$$\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

for any positive integer n.

Rational Functions

Main Strategy: Factor out the highest power of x in the denominator

Ex3/ For $f(x) = \dfrac{x^2-1}{2x^2+1}$, find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

$$f(x) = \frac{x^2-1}{2x^2+1} \overset{\text{When } x\neq 0}{=} \frac{\cancel{x^2}\left(\frac{x^2}{x^2} - \frac{1}{x^2}\right)}{\cancel{x^2}\left(\frac{2x^2}{x^2} + \frac{1}{x^2}\right)} = \frac{1-\frac{1}{x^2}}{2+\frac{1}{x^2}}$$

$$\boxed{(x^2+1) \quad x^3\left(\frac{x^2}{x^3} + \frac{1}{x^3}\right)}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1-\frac{1}{x^2}}{2+\frac{1}{x^2}} = \frac{1-0}{2+0} = \frac{1}{2}$$

$-\frac{1}{x^2} \to 0$, $+\frac{1}{x^2} \to 0$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1-\frac{1}{x^2}}{2+\frac{1}{x^2}} = \frac{1}{2}$$

Ex4/ Repeat for $f(x) = \dfrac{x^2+x}{-x+3} \underset{\text{when } x \text{ large}}{\approx} \dfrac{x^2}{-x} = -x$

$$f(x) = \frac{x^2+x}{-x+\frac{3}{x}\cdot x} \overset{x\neq 0}{=} \frac{x(x+1)}{x\left(-1+\frac{3}{x}\right)} = \frac{x+1}{-1+\frac{3}{x}}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+1}{-1+\frac{3}{x}} = -\infty$$

$x+1 \to +\infty$, $-1+\frac{3}{x} \to -1$

$$\lim_{x\to-\infty} \frac{x+1 \to -\infty}{-1+\frac{3}{x} \to -1} = \infty$$

Ex5/ Repeat for $f(x) = \dfrac{x^2+2}{x^3+x^2-1}$ When x large $\approx \dfrac{x^2}{x^3} = \dfrac{1}{x}$

$$f(x) = \frac{\frac{x^2 \cdot x}{x} + \frac{2x^3}{x^3}}{x^3 + \frac{x^2 \cdot x}{x} - \frac{1 \cdot x^3}{x^3}} = \frac{x^3\left(\frac{x^2}{x^3} + \frac{2}{x^3}\right)}{x^3\left(\frac{x^3}{x^3} + \frac{x^2}{x^3} - \frac{1}{x^3}\right)} = \frac{\frac{1}{x} + \frac{2}{x^3}}{1 + \frac{1}{x} - \frac{1}{x^3}}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{1}{x} + \frac{2}{x^3}}{1 + \frac{1}{x} - \frac{1}{x^3}} = \frac{0+0}{1+0-0} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\frac{1}{x} + \frac{2}{x^3}}{1 + \frac{1}{x} - \frac{1}{x^3}} = 0$$

Horizontal and Vertical Asymptote (HAs and VAs)

Ex6/ Find the locations of the HAs and VAs for $f(x) = \dfrac{3x^2+5}{x^2-4}$

VAs occur when the denominator is 0, but the numerator is non-zero.

$$\text{Denominator} = x^2 - 4 \overset{\text{set}}{=} 0$$

$$(x-2)(x+2) = 0$$

$$\boxed{x = -2, 2} \leftarrow \text{Location of VAs}$$

HAs occur when $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ is finite

$$\lim_{x\to\infty} \frac{3x^2+5}{x^2-4} = \lim_{x\to\infty} \frac{x^2\left(3+\frac{5}{x^2}\right)}{x^2\left(1-\frac{4}{x^2}\right)} = \lim_{x\to\infty} \frac{3+\frac{5}{x^2}}{1-\frac{4}{x^2}} = \frac{3+0}{1-0} = 3$$

NOTE Rational fcns can only have 1 HA

$y=3$ is a HA

Slant (Oblique) Asymptotes Recall polynomial long division

For a rational fcn $\frac{a(x)}{b(x)}$ there are polynomials $q(x)$ and $r(x)$ where

$$\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)} \quad ; \ \deg r < \deg b$$

$$\begin{array}{r} 2 \\ 2\overline{)5} \\ -4 \\ \hline 1 \end{array} \qquad \boxed{\frac{5}{2} = 2 + \frac{1}{2}}$$

Ex 7/

$$\begin{array}{r} 2x + 5 \\ x-1 \overline{) 2x^2 + 3x + 1} \\ -(2x^2 - 2x) \\ \hline 5x + 1 \\ -(5x - 5) \\ \hline 6 \end{array}$$

$\deg 6 < \deg (x-1)$ STOP

$$\frac{2x^2 + 3x + 1}{x-1} = \underbrace{2x+5} + \frac{6}{x-1}$$

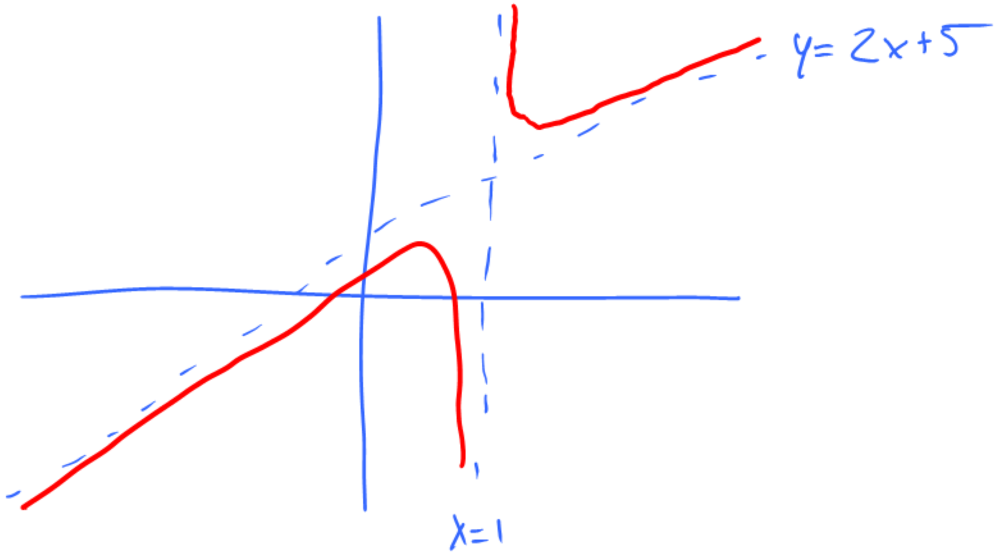
Theorem The following are equivalent

1. The fcn $\frac{a(x)}{b(x)}$ has a slant asymptote
2. The quotient in long division is a degree 1 polynomial
3. $\deg(a(x)) = \deg(b(x)) + 1$

Ex 8/

$$\frac{2x^2 + 3x + 1}{x-1} = \overbrace{(2x+5)}^{y=2x+5 \text{ is our slant asymptote}} + \frac{6}{x-1}$$

≈ 0 when x is large

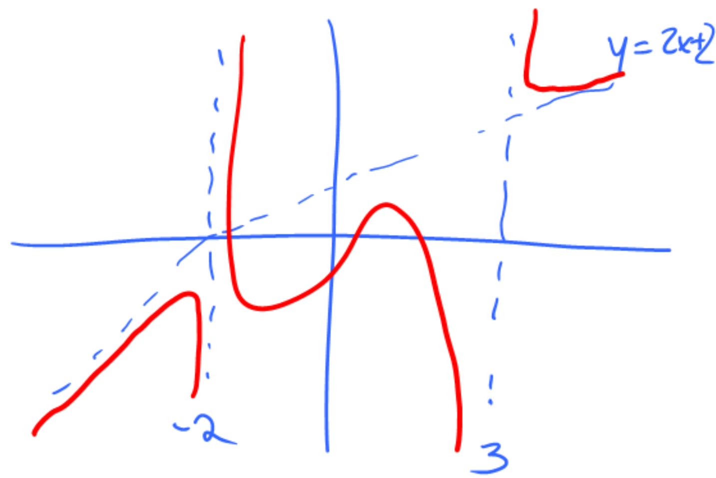


Ex/ Find Slant Asymptotes of $f(x) = \dfrac{2x^3-3x+1}{x^2-x-6}$

$$
\begin{array}{r}
2x+2 \\
x^2-x-6 \enclose{longdiv}{2x^3-3x+1} \\
-(2x^3-2x^2-12x) \\
\hline
2x^2+9x+1 \\
-(2x^2-2x-12) \\
\hline
11x+13
\end{array}
$$

Slant Asymptote:

$y = 2x+2$



f has no HAs