Lecture 22: Summary of Curve Sketching

Goal: Use the tools of calculus to draw a more accurate graph of a function.
All examples come from §4.5 of James Stewart's *Calculus Early Transcendentals*, 5th Edition as well as §4.4 of *Calculus Early Transcendentals*, 3rd Edition by Briggs et. al. (I'm too lazy to make my own examples).

All graphs will be available on Desmos (here).

Guidelines for Sketching a Curve

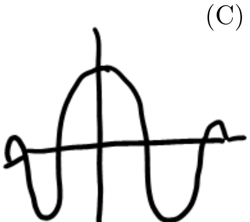
For drawing the graph of $y = f(x)$ by hand, consider:

(A) **Domain:** Find all values where f is defined.

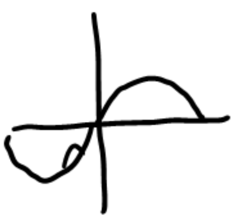
(B) **Intercepts:** Find the y-intercept by computing $f(0)$. Find the x-intercepts by solving the equation $f(x) = 0$.

(C) **Symmetry:**

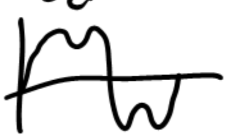
- (i) *Even functions:* When $f(x) = f(-x)$; this means the graph is symmetric about the y-axis.
- (ii) *Odd functions:* When $f(-x) = -f(x)$; this means it is symmetric about the origin (the left side is a 180° rotation of the right side)
- (iii) *Periodic functions:* When $f(x+p) = f(x)$ for some $p > 0$, the smallest such p is called the period. You only need to focus on one period of the function.



Even



Odd



(D) **Asymptotes and End Behavior:**

- (i) *Vertical Asymptotes:* Occurs at $x = a$ when either:

$$\lim_{x \to a^-} f(x) \text{ OR } \lim_{x \to a^+} f(x)$$

is not finite (equals ∞ or $-\infty$).

- (ii) *Horizontal Asymptotes:* Occurs at $y = L$ when either:

$$\lim_{x \to -\infty} f(x) = L \quad \text{OR} \quad \lim_{x \to \infty} f(x) = L$$

- (iii) *Slant Asymptotes:* The line $y = mx + b$ is a slant asymptote when either :

$$\lim_{x \to -\infty} [f(x) - (mx + b)] = 0 \quad \text{OR} \quad \lim_{x \to \infty} [f(x) - (mx + b)] = 0$$

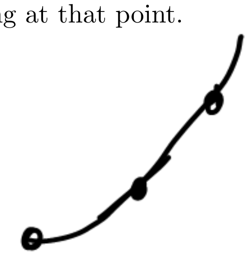
For rational functions, if the quotient from long division $q(x)$ is linear, then $q(x)$ is the slant asymptote.

(E) **Intervals of Increase or Decrease:** Use the I/D Test to see when f is increasing or decreasing. If $f' > 0$, then f is increasing. If $f' < 0$, f is decreasing.

(F) **Relative Maximum and Minimum Values:** Use the 1st or 2nd derivative test to determine if critical numbers are relative extrema.

(G) **Concavity and Inflection Points:** Use the concavity test. If $f'' > 0$, then f is concave upwards. If $f'' < 0$, f is concave downwards.

(H) **If needed, get more information:** You can also compute $(x, f(x))$ pairs to see the height of the graph. The tangent line also tells you where the graph is going at that point.



Ex/ Sketch a graph of $f(x) = \frac{1}{3}x^3 - 400x$

ⓐ Domain: $(-\infty, \infty)$

ⓑ Intercepts:

$f(0) = 0$

$(0,0)$ is our y-int

$\frac{1}{3}x^3 - 400x \overset{set}{=} 0$

$x^3 - 1200x = 0$

$x(x^2 - 1200) = 0$

$\Rightarrow x = 0, \pm\sqrt{1200} \approx \pm 34.6$

End Behavior:

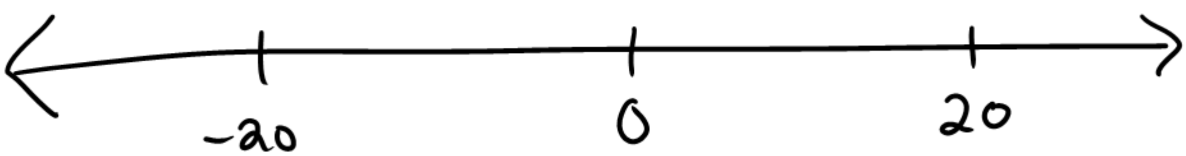
$\lim\limits_{x\to\infty}\left(\frac{1}{3}x^3 - 400x\right) = \infty$ $\quad \lim\limits_{x\to-\infty}\left(\frac{1}{3}x^3 - 400x\right) = -\infty$

Rel Max/Mins/ Inflection Points:

$f(x) = \frac{1}{3}x^3 - 400x$

$f'(x) = x^2 - 400 \overset{set}{=} 0 \Rightarrow x = \pm\sqrt{400} = \pm 20$

$f''(x) = 2x \overset{set}{=} 0 \Rightarrow x = 0$



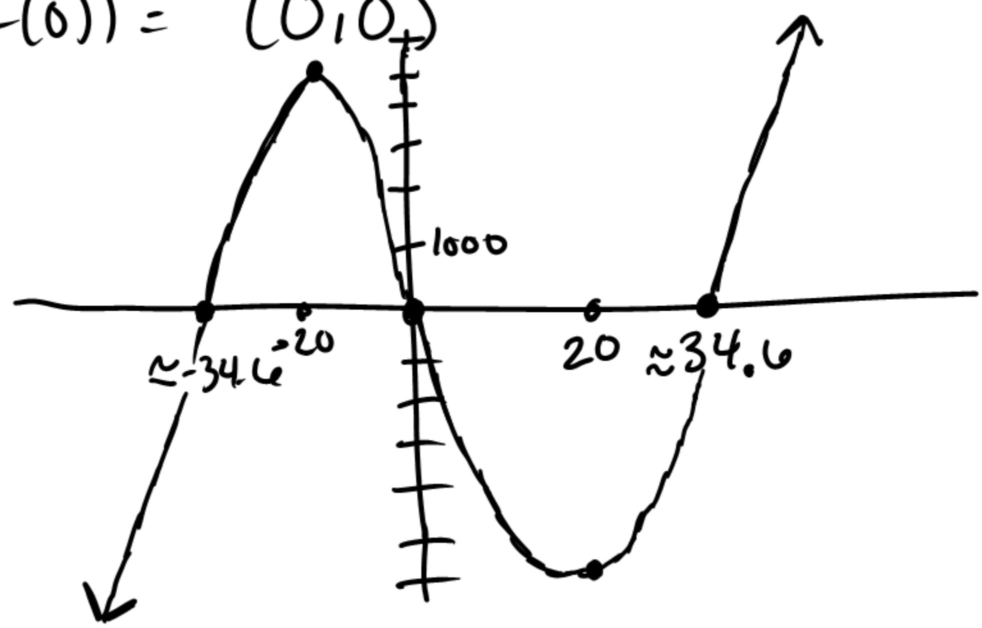
Test Points:	-100	-1	1	100
Sign of f':	+	−	−	+
Sign of f'':	−	−	+	+
Result I/D Test	Inc ↗	↘ Dec	Dec	Inc

Concavity	CD	CD	CU	CU

Rel Max: $(-20, f(-20)) = (-20, 5333 + \frac{1}{3})$

Rel Min: $(20, f(20)) = (20, -5333 + \frac{1}{3})$

Ip: $(0, f(0)) = (0,0)$



Ex 2/ Repeat for $f(x) = \frac{2x^2}{x^2-1}$

ⓐ Domain: Everything except $x^2 - 1 = 0 \Longrightarrow x = \pm 1$

$D = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

ⓑ Intercepts: $f(0) = 0$; so $(0,0)$ is both the x, y-int.

ⓒ $\lim\limits_{x \to \infty} \frac{2x^2}{x^2-1} = \lim\limits_{x \to \infty} \frac{2}{1 - \frac{1}{x^2}} = 2$. $y = 2$ is a H.A.

VAs: $x = \pm 1$

Using Calculus:

$$f(x) = \frac{2x^2}{x^2-1}$$

$$f'(x) = \frac{4x(x^2-1) - 2x^2(2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2} \overset{\text{set}}{=} 0$$

$\Rightarrow$ $x=0$ is our critical number

$$f''(x) = \frac{-4(x^2-1)^2 + 4x \cdot 2(x^2-1) \cdot 2x}{(x^2-1)^4} = \frac{12x^2+4}{(x^2-1)^3}$$

$12x^2+4$ is always positive. So,

$f'' > 0$ when $x^2-1 > 0$, or $x < -1$ OR $x > 1$

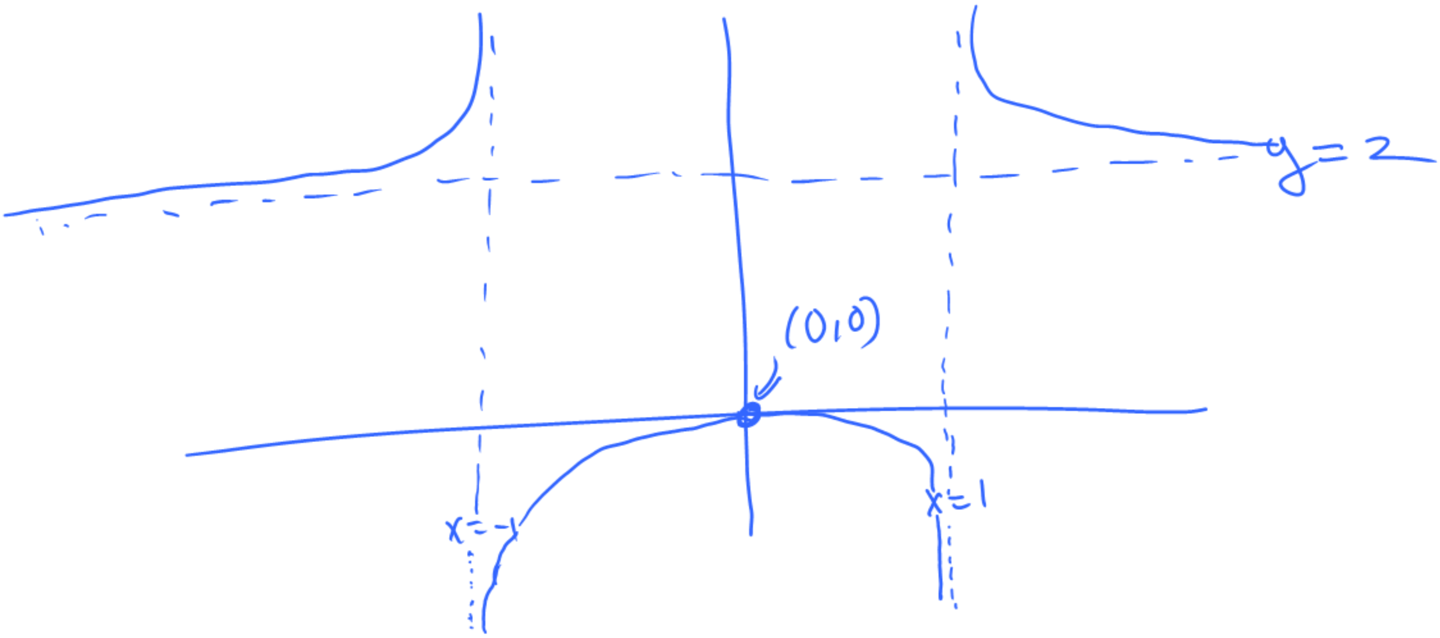
$f'' < 0$ when $x^2-1 < 0$, or $-1 < x < -1$

Use the sign chart for the sign of f'

$-1 \qquad 0 \qquad 1$

Test Points	-100	$-\frac{1}{2}$	$\frac{1}{2}$	100
Sign of $-4x$	+	+	−	−
Sign of $(x^2-1)^2$	+	+	+	+
Sign of f'	+	+	−	−

$\Rightarrow$ Rel Max at $x=0$. Max: $(0, f(0)) = (0,0)$



Ex4/ Repeat for $\dfrac{x^3}{x^2+1}$

ⓐ Domain: $(-\infty, \infty)$

ⓑ x-int: when $x^3=0 \Rightarrow x=0$

y-int: 0

ⓒ No VAs.

$$\lim_{x\to\infty}\left(\frac{x^3}{x^2+1}\right) = \lim_{x\to\infty}\frac{x^3}{x^2} = \infty$$

$$\lim_{x\to-\infty}\left(\frac{x^3}{x^2+1}\right) = -\infty$$

No HAs

Use long division to determine slant asymptote

$$\begin{array}{r} x \\ x^2+1 \overline{) x^3} \\ -(x^3+x) \\ \hline -x \end{array}$$

$y = x$ is a slant asymptote

$$f(x) = \frac{x^3}{x^2+1}$$

$$f'(x) = \frac{3x^2(x^2+1) - x^3 \cdot 2x}{(x^2+1)^2} = \frac{x^2(x^2+3)}{(x^2+1)^2} \overset{\text{set}}{=} 0$$

$$\Rightarrow x^2(x^2+3) = 0 \Rightarrow x = 0$$

Test Point	-100	0	100
f'	$+$		$+$

f is always increasing

$$f''(x) = \frac{2x(3-x^2)}{(x^2+1)^3} \overset{\text{set}}{=} 0$$

$\Rightarrow x = 0, \pm\sqrt{3}$ are potential inflection points

		$-\sqrt{3}$		0		$\sqrt{3}$	
Sign of f'':	$+$ CU		$-$ CD		$+$ CU		$-$ CD