Lecture 22: Summary of Curve Sketching

Goal: Use the tools of calculus to draw a more accurate graph of a function.
All examples come from §4.5 of James Stewart's *Calculus Early Transcendentals*, 5th Edition as well as §4.4 of *Calculus Early Transcendentals*, 3rd Edition by Briggs et. al. (I'm too lazy to make my own examples).

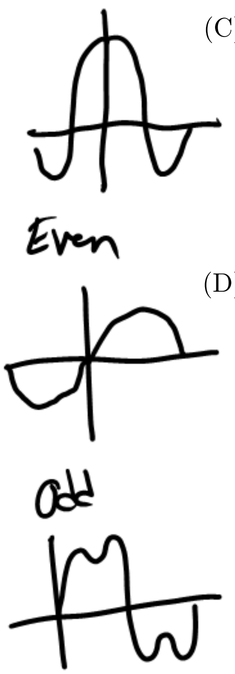
All graphs will be available on Desmos (here).

Guidelines for Sketching a Curve

For drawing the graph of $y = f(x)$ by hand, consider:

(A) **Domain:** Find all values where f is defined.

(B) **Intercepts:** Find the y-intercept by computing $f(0)$. Find the x-intercepts by solving the equation $f(x) = 0$.



(C) **Symmetry:**

  (i) *Even functions:* When $f(x) = f(-x)$; this means the graph is symmetric about the y-axis.

  (ii) *Odd functions:* When $f(-x) = -f(x)$; this means it is symmetric about the origin (the left side is a 180° rotation of the right side)

  (iii) *Periodic functions:* When $f(x+p) = f(x)$ for some $p > 0$, the smallest such p is called the period. You only need to focus on one period of the function.

(D) **Asymptotes and End Behavior:**

  (i) *Vertical Asymptotes:* Occurs at $x = a$ when either:

$$\lim_{x \to a^-} f(x) \text{ OR } \lim_{x \to a^+} f(x)$$

  is not finite (equals ∞ or $-\infty$).

  (ii) *Horizontal Asymptotes:* Occurs at $y = L$ when either:

$$\lim_{x \to -\infty} f(x) = L \quad \text{OR} \quad \lim_{x \to \infty} f(x) = L$$

  (iii) *Slant Asymptotes:* The line $y = mx + b$ is a slant asymptote when either :

$$\lim_{x \to -\infty} [f(x) - (mx + b)] = 0 \quad \text{OR} \quad \lim_{x \to \infty} [f(x) - (mx + b)] = 0$$

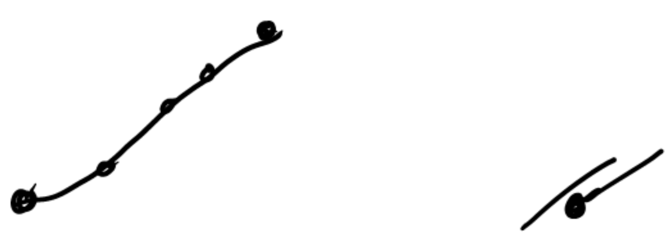
  For rational functions, if the quotient from long division $q(x)$ is linear, then $q(x)$ is the slant asymptote.

(E) **Intervals of Increase or Decrease:** Use the I/D Test to see when f is increasing or decreasing. If $f' > 0$, then f is increasing. If $f' < 0$, f is decreasing.

(F) **Relative Maximum and Minimum Values:** Use the 1st or 2nd derivative test to determine if critical numbers are relative extrema.

(G) **Concavity and Inflection Points:** Use the concavity test. If $f'' > 0$, then f is concave upwards. If $f'' < 0$, f is concave downwards.

(H) **If needed, get more information:** You can also compute $(x, f(x))$ pairs to see the height of the graph. The tangent line also tells you where the graph is going at that point.



Ex1/ Sketch a graph of $f(x) = \frac{1}{3}x^3 - 400x$

Domain: $(-\infty, \infty)$

Intercepts: y-int: $f(0) = 0$. Passes through $(0,0)$

x-int: $\frac{1}{3}x^3 - 400x \overset{\text{set}}{=} 0$

$$x^3 - 1200x = 0$$

$$x(x^2 - 1200) = 0$$

$\Rightarrow$ x-ints are $x=0$ and $x = \pm\sqrt{1200} \approx \pm 34.6$

End Behaviors:

$$\lim_{x\to\infty}\left(\frac{1}{3}x^3 - 400x\right) = \lim_{x\to\infty}\frac{1}{3}x^3 = \infty$$

$$\lim_{x\to-\infty}\left(\frac{1}{3}x^3 - 400x\right) = \lim_{x\to-\infty}\frac{1}{3}x^3 = -\infty$$

Rel Max/Mins/ IPs:

$$f(x) = \frac{1}{3}x^3 - 400x$$

$$f'(x) = x^2 - 400 \overset{\text{set}}{=} 0 \Rightarrow x = \pm\sqrt{400} = \pm 20$$

$$f''(x) = 2x \overset{\text{set}}{=} 0 \Rightarrow x = 0$$

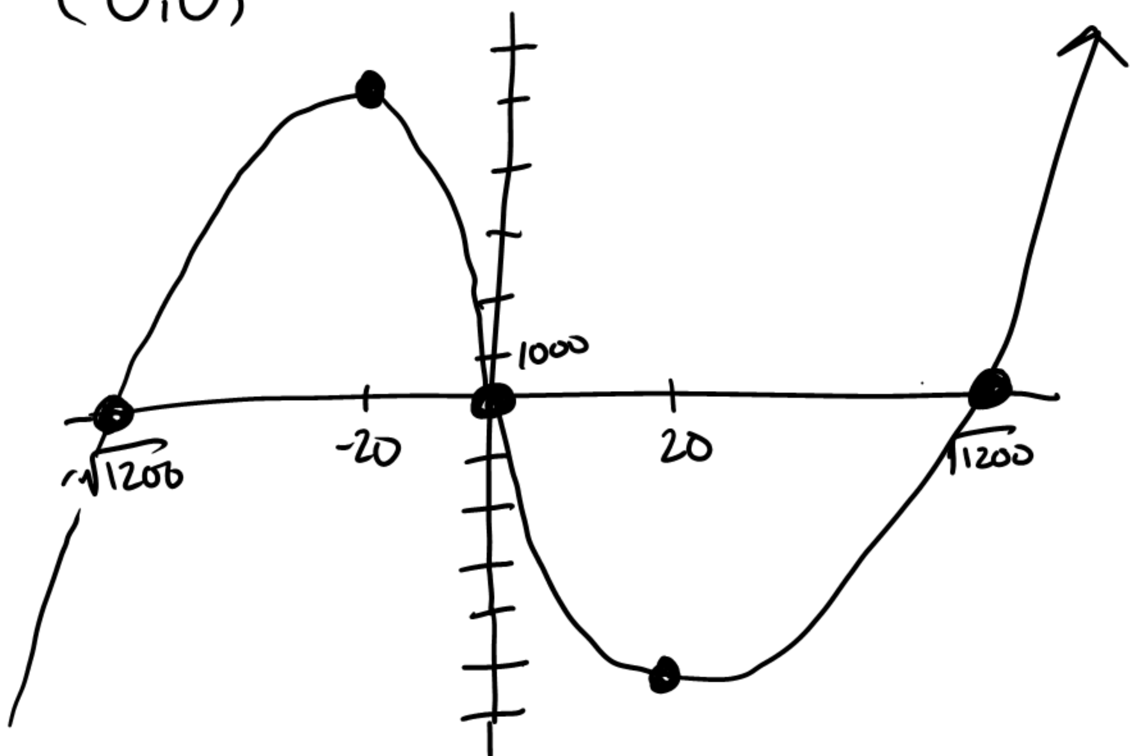
		-20		0		20	
Test Point:	-100		-1		1		100
Sign of f' :	$+$		$-$		$-$		$+$
Sign of f'' :	$-$		$-$		$+$		$+$

Result of I/D Test:	Inc ↗	↘ Dec	Dec ↘	↗ Inc
Concavity Test:	CD	CD	CU	CU

Rel Max: $(-20, f(-20)) = (-20, 5333 + \frac{1}{3})$

Rel Min: $(20, f(20)) = (20, -5333 + \frac{1}{3})$

Inf Point: $(0,0)$



Ex3/ Repeat for $f(x) = \dfrac{2x^2}{x^2-1}$

ⓐ Domain: All but ± 1

ⓑ Intercepts: y-int: $f(0) = 0$ $(0,0)$

x-int: $2x^2 \underline{\underline{\text{set}}} \, 0 \Rightarrow x = 0$

ⓓ Asymptotes: VAs when $x^2 - 1 = 0 \Longrightarrow \boxed{x = \pm 1}$

HAs:

$$\lim_{x\to\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\infty} \frac{2}{1-\frac{1}{x^2}} = 2$$

Likewise taking $x \to -\infty$ gets us 2. $\boxed{y=2 \text{ is a HA}}$

$$f(x) = \frac{2x^2}{x^2-1}$$

$$f'(x) = \frac{4x(x^2-1) - 2x^2(2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2} \overset{\text{set}}{=} 0$$

$\Rightarrow -4x = 0 \Rightarrow x = 0$ is a crit number

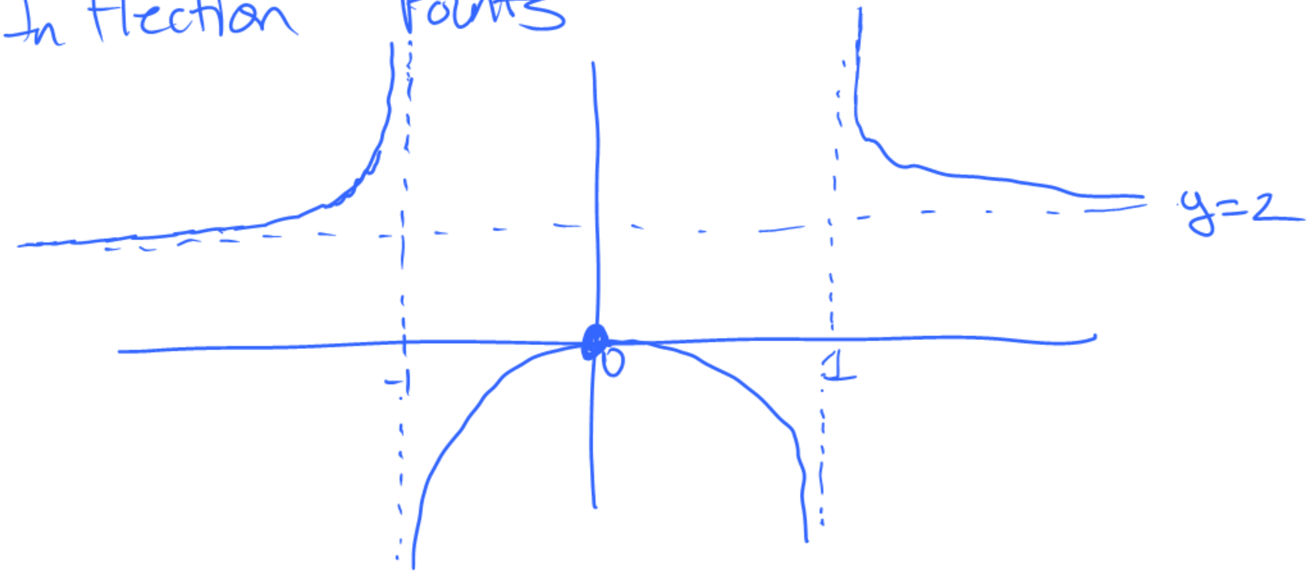
$$f''(x) = \frac{-4(x^2-1)^2 + 4x \cdot 2(x^2-1) \cdot 2x}{(x^2-1)^4} = \frac{12x^2+4}{(x^2-1)^3}$$

Test Value	-100	$-\frac{1}{2}$	$\frac{1}{2}$	100
Sign of f' = Sign of $-4x$	+	+	−	−
Sign of $12x^2+4$	+	+	+	+
Sign of x^2-1	+	−	−	+
Sign of $(x^2-1)^3$	+	−	−	+
Sign of f''	+	−	−	+
Inc / Dec	Inc	Inc	Dec	Dec

Concavity | CU | CD | CD | CU

Rel Max @ $(0, f(0)) = (0,0)$

No Inflection Points



Ex 4/ Repeat for $f(x) = \dfrac{x^3}{x^2+1}$

Domain: $(-\infty, \infty)$

Intercepts: $(0,0)$

Asym. and End Behavior: No VAs, No HAs, There is a slant asymptote

$$\begin{array}{r} x \\ x^2+1 \overline{)\, x^3} \\ -(x^3+x) \\ \hline -x \end{array}$$

$y = x$ is our slant asymptote

$$f(x) = \frac{x^3}{x^2+1}$$

$$f'(x) = \frac{3x^2(x^2+1) - x^3(2x)}{(x^2+1)^2} = \frac{\overbrace{x^2}^{\geq 0}\overbrace{(x^2+3)}^{>0}}{\underbrace{(x^2+1)^2}_{>0}} \overset{\text{set}}{=} 0$$

$$\Rightarrow x^2(x^2+3)=0 \Rightarrow \boxed{x=0}$$

f is always increasing by I/D Test

$$f''(x)=\frac{2x(3-x^2)}{(x^2+1)^3} \overset{\text{set}}{=} 0$$

$$\Rightarrow 2x(3-x^2)=0 \Rightarrow x=0,\ \pm\sqrt{3}$$

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
Test Points	-100	-1	1	100
Sign of f''	$+$	$-$	$+$	$-$
Con. Test	CU	CD	CU	CD

The points $\left(\pm\sqrt{3}, \pm\frac{3\sqrt{3}}{4}\right)$ and $(0,0)$ are inflection points