Goal: Use the tools to find maximums and minimums in applications.

<u>Idea</u> Given a situation, find the "optimial" value (usually a max/min)

Ex/ A farmer is building a rectangular fence and has <u>400 ft</u> worth of fencing to work with. What should the dimensions of the fence so the area is as large as possible?

The function we want optimize is called the "<u>objective</u>" fcn

$$A(l, w) = lw$$

To reduce the number of variables and/or guarantee a max/min we need another condition, called the <u>constraint</u>.

$$P = \boxed{2l + 2w = 400}$$

We can rewrite

Obj: Maximize $A_{(l,w)} = lw$

Given: $2l + 2w = 400$

Rewrite A solely as a fcn of w:

$$2l = 400 - 2w$$

$$l = \frac{400 - 2w}{2} = 200 - w$$

$$A_{(w)} = (200-w)w = 200w - w^2$$

Find an absolute max:

$$\frac{dA}{dw} = 200 - 2w \stackrel{set}{=} 0$$

$$200 = 2w$$

$$\boxed{w = 100}$$

Verify that $w=100$ is the location of the max:

$$\frac{d^2A}{dw^2} = -2 < 0$$

Local Max + Only 1 crit. value $\Longrightarrow$ Absolute Maximum

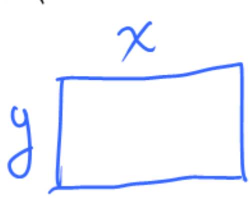
Answer the question:

$$l = 200 - w$$

$$l\Big]_{100} = 200 - 100 = 100$$

Conclusion: To maximize area, the fence needs to be a square with side length 100 ft

Ex 2/ Suppose two numbers x and y add to $S>0$. When is their product at a maximum?



$x + \underbrace{y = S}_{}$ constant

$2x + 2y = 2S$

Obj: Maximize $P_{(x,y)} = xy$

Given: $x + y = S$

$\Rightarrow \quad y = S - x$. So,

$$P_{(x)} = x(S-x) = Sx - x^2$$

Find Abs. Max:

$$\frac{dP}{dx} = S - 2x \overset{\text{set}}{=} 0$$

$$x = \frac{S}{2}$$

Verify its the loc of the max:

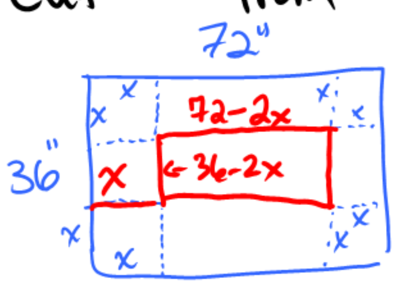
$$\frac{d^2P}{dx^2} = -2 < 0$$

Local Max by the 2^{nd} Derivative Test
+
Only 1 Crit. Value
$\Longrightarrow$ Absolute Max

$$y = S - x \Rightarrow y\Big]_{x=\frac{S}{2}} = S - \frac{S}{2} = \frac{S}{2}$$

Conclusion: If $x+y=S$, then their product xy is maximized when $x = y = \frac{S}{2}$

Ex 3/ A 36" x 72" piece of cardboard is cut from each corner and the sides folded up to create an open top box. How far should be cut from the edge to maximize volume?



Obj: Maximize $V_{(x)} = (72-2x)(36-2x)x$

Given: $x \in [0, 18]$

$$V(x) = 2592x - 216x^2 + 4x^3$$

$$\frac{dV}{dx} = 2592 - 432x + 12x^2 \overset{\text{set}}{=} 0$$

Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ given $ax^2 + bx + c = 0$

$$x = 18 \pm 6\sqrt{3} \xrightarrow{18 + 6\sqrt{3} \notin [0,18]} x = 18 - 6\sqrt{3} \approx 7.61$$

We are on a closed interval

x	$18-6\sqrt{3}$	0	18
$V(x)$	>0	0	0

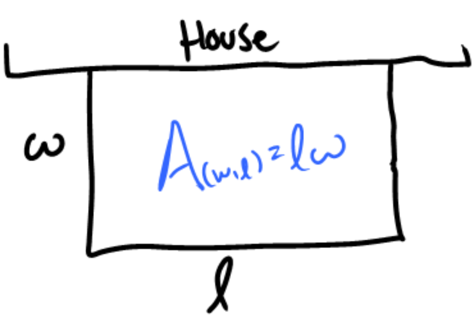
Abs. Max at $18 - 6\sqrt{3}$

$50' = 50$ ft

$50'' = 50$ inches

Conclusion: We need to cut ≈ 7.61 in away from the edge to maximize volume.

Ex 4/ You are making a rectangular garden and you have 50' of fencing to work with. To make a bigger garden, you build it right next to your house. Find the maximum area?



Obj: Maximize $A_{(l,w)} = lw$

Given: $2w + l = 50$

$$\Rightarrow l = 50 - 2w$$

$$A_{(w)} = (50 - 2w)w = 50w - 2w^2$$

$$\frac{dA}{dw} = 50 - 4w \overset{\text{set}}{=} 0$$

$$w = \frac{50}{4} = 12.5 = \frac{25}{2}$$

Verify it is a max:

$$\frac{d^2A}{dw^2} = -4 < 0 \Rightarrow \text{Local Max by the } 2^{nd} \text{ Derivative Test}$$

Local Max + 1 crit value $\Rightarrow$ Abs. Max

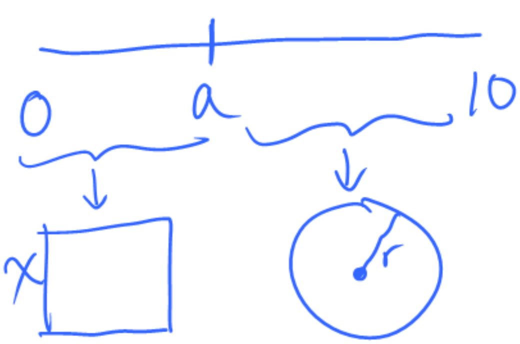
Find area: Given $w = \frac{25}{2}$

$$l\Big]_{w=\frac{25}{2}} = 50 - 2\left(\frac{25}{2}\right) = 25$$

$$A = lw \Rightarrow A\Big]_{l=25}^{w=25/2} = 25\left(\frac{25}{2}\right) = \left(312 + \frac{1}{2}\right) ft^2.$$

Conc: The maximal area is $312.5\ ft^2$.

Ex/ A 10m is wire is cut into 2 pieces. One of them is bent into a square, the other is bent into a circle. Where should we cut so that the enclosed area is maximal



Obj: Maximize $A_{(x,r)} = x^2 + \pi r^2$

Given: $\begin{cases} 4x = a;\ 2\pi r = 10 - a \Rightarrow \boxed{4x + 2\pi r = 10} \\ a \in [0, 10] \Rightarrow \boxed{x \in [0, 2.5]} \end{cases}$

$$4x + 2\pi r = 10$$

$$2x + \pi r = 5$$

$$r = \frac{1}{\pi}(5 - 2x)$$

$$A_{(x)} = x^2 + \pi\left(\frac{1}{\pi}[5-2x]\right)^2$$
$$= \left(1+\frac{4}{\pi}\right)x^2 - \frac{20}{\pi}x + \frac{5}{\pi}$$

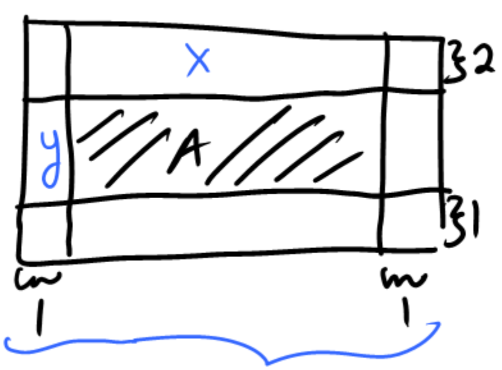
$$\frac{dA}{dx} = 2\left(1+\frac{4}{\pi}\right)x - \frac{20}{\pi} \overset{\text{set}}{=} 0$$
$$x = \frac{10}{\pi+4} \approx 1.4 \quad ; x \in [0, 2.5]$$

x	$\frac{10}{\pi+4}$	0	2.5
$A(x)$	≈ 3.5	≈ 7.95	$= 6.25$

Maximized when $x = 0 \Rightarrow a = 0$

Ie, we don't cut it at all

Ex6/ A poster board has an area of $180\ cm^2$ and has margins like the diagram. What is the largest printable area?



Obj: Maximize $A = xy$

Given: $(x+1+1)(y+1+2) = 180$

$$\Rightarrow y = \frac{180}{x+2} - 3$$

$\boxed{\text{See 8:30 Section Notes for Solution}}$