**Goal:** Use the tools to find maximums and minimums in applications.

Idea: Given a situation, find the "optimal" value (usually a max/min)

400′ = 400 ft ; 400″ = 400 in

Ex/ A farmer is building a rectangular pig pen and has 400′ of fencing to work with. What should the dimensions of the pen be so that the area is as large as possible?


The function we want to optimize is called the objective function

$$A(l,w) = lw$$

To reduce the # of variables or to guarantee a max/min we need another condition, called the constraint

$$2l + 2w = 400$$

Rewriting this: Obj: Maximize $A(l,w) = lw$

Given: $2l + 2w = 400$

Rewrite $A$ solely as a fcn of $w$:

$$2l = 400 - 2w$$

$$l = \frac{400 - 2w}{2} = 200 - w$$

$$A(w) = (200 - w)w = 200w - w^2$$

We now want to find an abs. max

$$\frac{dA}{dw} = 200 - 2w \overset{\text{set}}{=} 0$$
$$200 = 2w$$
$$w = 100$$

Verify $w=100$ is the location of the maximum.

$$\frac{d^2A}{dw^2} = -2 < 0 \Rightarrow \text{Local Max at } w=100$$

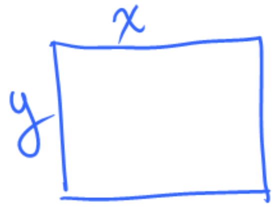
Local Max + Fcn has only 1 crit. value $\Longrightarrow$ Abs. Max

Answer the question: Width $= 100'$

$$l = 200 - w$$
$$l\Big]_{w=100} = 200 - 100 = 100$$

The area of the pig pen is maximized when it is a square with side length 100 ft.

Ex 3) Suppose two numbers $x$ and $y$ add up to $S>0$. When is their product maximized?




$x+y=S$

$2x+2y=2S$

Obj: Maximize $P_{(x,y)} = xy$

Given: $x+y=S \Rightarrow y = S-x$

$$P_{(x)} = x(S-x) = Sx - x^2$$

$$\frac{dP}{dx} = S - 2x \overset{\text{set}}{=} 0 \Longrightarrow x = \frac{S}{2}$$

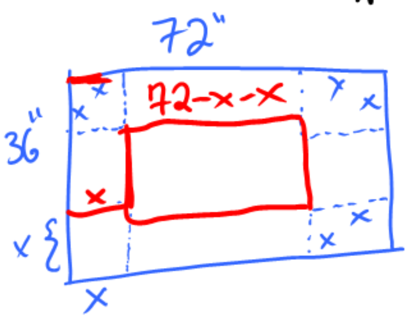
Verify it's a max: $\frac{d^2P}{dx^2} = -2 < 0$, local max by $2^{nd}$ $\frac{d}{dx}$ test

Local Max + Only 1 crit. num. $\Rightarrow$ Abs. Max

$$y = S - x \Rightarrow y\Big]_{x=\frac{S}{2}} = S - \frac{S}{2} = \frac{S}{2}$$

Conclusion: Given $x + y = S$, the product $xy$ is maximized when $x = y = \frac{S}{2}$

Ex3/ A 36"x72" piece is cardboard is cut from each corner and the sides are folded up to create an open-top box. How far should we cut from the edge to max. volume?




Obj: Maximize $V_{(x)} = (72-2x)(36-2x)x$

Given: $x \in [0, 18]$

$$V(x) = 2592x - 216x^2 + 4x^3$$

$$\frac{dV}{dx} = 2592 - 432x + 12x^2 \underset{=}{\text{set}} 0$$

Quadratic Formula: $ax^2 + bx + c = 0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

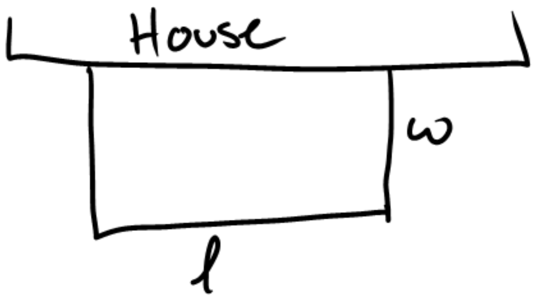
$$\Rightarrow x = 18 \pm 6\sqrt{3} \xrightarrow{18+6\sqrt{3} \notin [0,18]} x = 18 - 6\sqrt{3} \approx 7.61$$

↙ Loc. of Max

| $x$ | $18-6\sqrt{3}$ | $0$ | $18$ |
|---|---|---|---|
| $V(x)$ | $>0$ | $0$ | $0$ |

Ie, we need to cut $\approx 7.61$in away from the edge to maximize volume.

Ex 4/ You are building a rectangular garden and have 50' of fencing to work with. To make a bigger garden, you decide to build it next to your house. Find the max. area.




Obj: Maximize $A_{(l,w)} = lw$

Given: $2w + l = 50$

$l = 50 - 2w$

$$A_{(w)} = (50-2w)w = 50w - 2w^2$$

$$\frac{dA}{dw} = 50 - 4w \overset{\text{set}}{=} 0$$

$$w = \frac{50}{4} = \frac{25}{2} = 12.5$$

Verify it's a max: $\frac{d^2A}{dw^2} = -4 < 0$ Local Max + Only 1 crit. num. $\Rightarrow$ Global Max

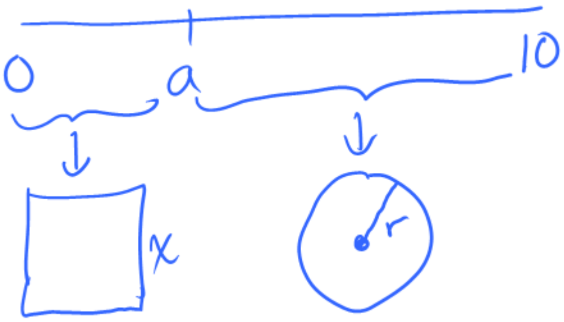
Given $w = \frac{25}{2}$

$$l = 50 - 2w \Rightarrow l\Big]_{w=\frac{25}{2}} = 50 - 2\left(\frac{25}{2}\right) = 25$$

$$A = lw \Rightarrow A\Big]_{l=25}^{w=25/2} = 25\left(\frac{25}{2}\right) = \left(312 + \frac{1}{2}\right) ft^2$$

The maximum area is $312.5\ ft^2$.

Ex 5/ A 10m wire is cut into 2 pieces. One piece is bent into a square and the other is bent into a circle. Where should we cut so that the enclosed area is maximal?




$$A = x^2 + \pi r^2$$

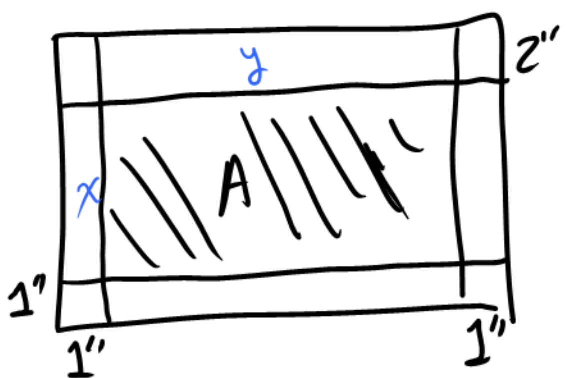
$$\begin{cases} 4x = a \Rightarrow x = \frac{a}{4} \\ 2\pi r = 10 - a \Rightarrow r = \frac{10-a}{2\pi} \end{cases}$$

Obj: $A_{(a)} = \left(\frac{a}{4}\right)^2 + \pi\left(\frac{10-a}{2\pi}\right)^2$

Given: $a \in [0, 10]$

See 7:30 Section's Notes for Solution

Ex 6/ A poster board is to have an area of 180 in$^2$ with margins like the diagram below. What is the largest possible printing area?




Obj: Maximize $A = xy$

Given: $(x+3)(y+2) = 180$ and $x>0$, $y>0$

$$y + 2 = \frac{180}{x+3}$$

$$y = \frac{180}{x+3} - 2$$

Reminder: This is our objective for $xy$, not the constraint $(x+3)(y+2)$

$$A_{(x)} = x\left(\frac{180}{x+3} - 2\right) = \frac{180x}{x+3} - 2x$$

$$\frac{dA}{dx} = \frac{180(x+3) - 180x}{(x+3)^2} - 2 \overset{\text{set}}{=} 0$$

$$\frac{540}{(x+3)^2} - 2 = 0$$

$$\frac{540}{(x+3)^2} = 2$$

$$\frac{270}{(x+3)^2} = 1$$

$$270 = (x+3)^2$$

$$x^2 + 6x - 261 = 0$$

$$\Rightarrow x = -3 \pm 3\sqrt{30} \xRightarrow{x>0} x = -3 + 3\sqrt{30} \approx 13.43$$

It is indeed a max

$$y = \frac{180}{(-3+3\sqrt{30})+3} - 2 = \frac{60}{\sqrt{30}} - 2 = 2\sqrt{30} - 2$$

Area: $(-3 + 3\sqrt{30})(2\sqrt{30} - 2)\,in^2$