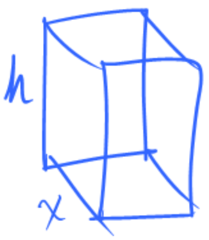
Goal: Solve optimization problems involving volume, surface area, and distances.

Ex/ We want to construct a box with a square base and no-top. If the volume is 500 in^3, then what is the minimum amount of material required?



Obj: Minimize $A_{(x,h)} = x^2 + 4xh$

Given: $500 = x^2 h$; $x, h > 0$

$$h = \frac{500}{x^2}$$

$$A_{(x)} = x^2 + 4x\left(\frac{500}{x^2}\right) = x^2 + \frac{2000}{x}$$

$$\frac{dA}{dx} = 2x - \frac{2000}{x^2} \overset{\text{set}}{=} 0$$

$$2x = \frac{2000}{x^2}$$

$$x = \frac{1000}{x^2}$$

$$\sqrt[3]{x^3} = \sqrt[3]{1000} \longrightarrow x = 10$$

Verify it's a minimum:

Local Min by 1st $\frac{d}{dx}$ test
+
Only one crit. num. $\Longrightarrow$ Abs. Min

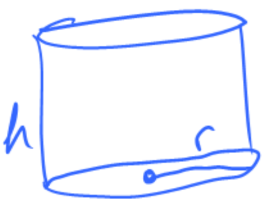
$x = 10$

$$h = \frac{500}{10^2} = \frac{500}{100} = 5$$

$$A_{(10,5)} = 10^2 + 4(10)(5) = 300$$

Conclusion: It only takes 300 in^2 worth of materials to produce such a box.

Ex 2 (Ideal Soup Can) A company is designing a cylindrical soup can to hold 250π (≈ 785.4) cm^3 of liquid. What are the dimensions that minimize costs?



Obj: Minimize $A_{(r,h)} = 2\pi r^2 + 2\pi r h$

Given: $250\pi = \pi r^2 h \; ; r,h > 0$

$$h = \frac{250\pi}{\pi r^2} = \frac{250}{r^2}$$

$$A_{(r)} = 2\pi r^2 + 2\pi r\left(\frac{250}{r^2}\right) = 2\pi r^2 + \frac{500\pi}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{500\pi}{r^2} \stackrel{\text{set}}{=} 0$$

$$4\pi r = \frac{500\pi}{r^2}$$

$$r = \frac{125}{r^2}$$

$$r^3 = 125 \longrightarrow r = 5$$

You can verify it is indeed a minimum

$$r = 5$$

$$h = \frac{250}{r^2} \Rightarrow h\Big]_{r=5} = \frac{250}{5^2} = \frac{250}{25} = 10$$

Conclusion: The radius needs to be 5 cm while the height needs to be 10 cm to minimize costs.

Ex3 (Ideal Soup Can II)

Due to the manufacturing process each can has to use 294π (≈ 923.63) cm^2 of aluminium.
Find the maximal volume?

Obj: Maximize $V_{(r,h)} = \pi r^2 h$

Given: $294\pi = 2\pi r^2 + 2\pi r h$

$$2\pi r h = 294\pi - 2\pi r^2$$

$$h = \frac{294\pi - 2\pi r^2}{2\pi r}$$

$$h = \frac{147}{r} - r$$

$$V_{(r)} = \pi r^2\left(\frac{147}{r} - r\right) = 147\pi r - \pi r^3$$

$$\frac{dV}{dr} = 147\pi - 3\pi r^2 \underset{=}{\text{set}} 0$$

$$3\pi r^2 = 147\pi$$

$$r^2 = \frac{147\pi}{3\pi}$$

$$r^2 = 49 \longrightarrow \boxed{r = 7}$$

Verify it's a max:

$$\frac{d^2V}{dr^2} = -9\pi r$$

$$\left.\frac{d^2V}{dr^2}\right]_{r=7} = -63\pi < 0$$

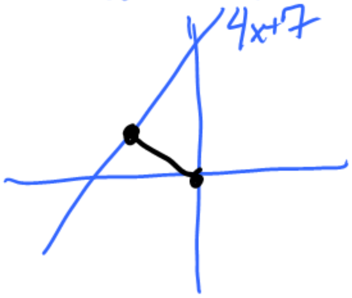
Local Max by 2^{nd} $\frac{d}{dx}$ test $\Rightarrow$ Abs. Max.
+ Only 1 Crit. Num

$$V(7) = \pi \cdot 7^2\left(\frac{147}{7} - 7\right) = 686\pi$$
$$\approx 2155.13$$

Conclusion: The maximum volume is roughly $2{,}155.13 \text{ cm}^3$

Distances

Ex4 What is the smallest distance between $y = 4x+7$ and the origin



Recall the distance between (x,y) and (x_0, y_0) is

$$d = \sqrt{(x - \underbrace{x_0}_{=0})^2 + (y - \underbrace{y_0}_{=0})^2}$$

Fact: Finding the location where the optimal distance occurs is the same as finding the location where the optimal squared distance occurs.

$$(\text{max})^2 \geq (\text{The rest})^2$$
$$\sqrt{(\text{max})^2} \geq \sqrt{(\text{The rest})^2}$$
$$\text{max} \geq \text{The rest}$$

B/c distances are positive

Back to ex Obj: Minimize $S(x,y) = d^2(x,y) = x^2 + y^2$

Given: $y = 4x+7$

$$S(x) = x^2 + (4x+7)^2 = x^2 + 16x^2 + 56x + 28$$

$$S(x) = 17x^2 + 56x + 28$$

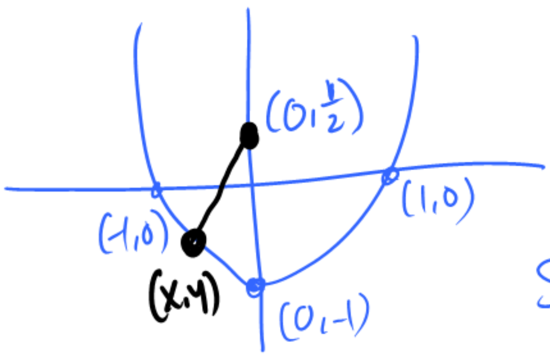
$$\frac{dS}{dx} = 34x + 56 \overset{\text{set}}{=} 0$$

$$x = \frac{-56}{34} = \frac{-28}{17}$$

Y-coord: $y = 4\left(\frac{-28}{17}\right) + 7 = \frac{7}{17}$

Min. Dist: $\sqrt{\left(\frac{-28}{17}\right)^2 + \left(\frac{7}{17}\right)^2} = \frac{7}{\sqrt{17}}$

Ex 5 Where is the distance between $(0, \frac{1}{2})$ and the parabola $y = x^2 - 1$ at a minimum?



Obj: Minimize $S(x,y) = d^2(x,y) = x^2 + \left(y - \frac{1}{2}\right)^2$

Given: $y = x^2 - 1$

$$S(x) = x^2 + \left(x^2 - 1 - \frac{1}{2}\right)^2 = x^2 + \left(x^2 - \frac{3}{2}\right)^2$$

$$= x^2 + x^4 - 3x^2 + \frac{9}{4}$$

$$= x^4 - 2x^2 + \frac{9}{4}$$

$$\frac{dS}{dx} = 4x^3 - 4x \overset{\text{set}}{=} 0$$

$$4x(x^2 - 1) = 0$$

$$4x(x-1)(x+1) = 0$$

$x = \pm 1, 0$

See which ones are local mins

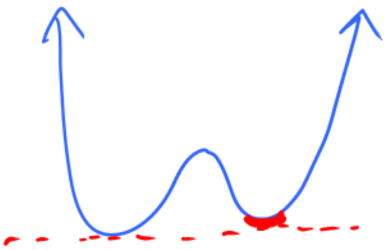
$$\frac{d^2S}{dx^2} = 12x^2 - 4$$

$$\left.\frac{d^2S}{dx^2}\right]_{x=-1} = 8 > 0 \qquad \left.\frac{d^2S}{dx^2}\right]_{x=1} = 8 > 0$$

$$\left.\frac{d^2S}{dx^2}\right]_{x=0} = -4 < 0$$

By the 2^{nd} $\frac{d}{dx}$ test, $x = \pm 1$ are the locs. of <u>local mins</u>. But does a global min exist?

Looking at end behavior $S \to +\infty$ as $x \to \pm\infty$



$\Leftarrow$ A global min exists, but not a global max

NOTE: Since $S(1) = S(-1)$, the global min occurs at both places.

Ie, the minimum distance occurs at $(\pm 1, 0)$.