Goal: Solve optimization problems involving cost, revenue, and profit.

Costs can be incorporated in two ways

(1) As the objective fcn

Ex) Jessica wants to make a box such that

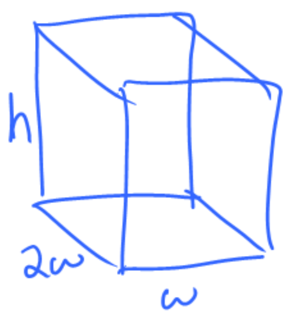
(i) The volume is $144 ft^3$

(ii) The length is double the width

(iii) The top and bottom are made out of metal

(iv) The sides are made out of wood.

If it costs \$10/ft² for wood and \$20/ft² for metal, what should the dimensions be to make this box as cheaply as possible?



$$\text{Cost} = \left(\begin{matrix}\text{Price of}\\ \text{Wood}\end{matrix}\right) + \left(\begin{matrix}\text{Price of}\\ \text{Metal}\end{matrix}\right)$$

$$= \left(\frac{\text{Price}}{\text{Sq. ft}}\right)\left(\#\ \text{sq. ft}\right) + \left(\frac{\text{Price}}{\text{Sq. ft}}\right)\left(\begin{matrix}\text{Sq. ft.}\\ \text{of metal}\end{matrix}\right)$$

$$= 10\left(2wh + 2(2wh)\right) + 20\left((2w\cdot w)2\right)$$

$$C_{(w,h)} = 60hw + 80w^2$$

Obj: Minimize $C_{(w,h)} = 60hw + 80w^2$

Given: $144 = w(2w)h = 2hw^2$

$$\Rightarrow h = \frac{144}{2w^2} = \frac{72}{w^2}$$

$$C_{(w)} = 60\left(\frac{72}{w^2}\right)w + 80w^2 = \frac{4320}{w} + 80w^2$$

$$\frac{dC}{dw} = -\frac{4320}{w^2} + 160w \overset{\text{set}}{=} 0$$

$$160w = \frac{4320}{w^2}$$

$$160w^3 = 4320$$

$$w^3 = \frac{4320}{160} = 27$$

$$\boxed{w = 3}$$

Verify it is a max: 

Local Min + 1 Crit. Num $\Rightarrow$ Abs. Min

Length: $2w \Rightarrow$ length: $2(3) = 6$

Height: $h = \frac{72}{w^2} \Rightarrow h\Big]_{w=3} = \frac{72}{3^2} = \frac{72}{9} = 8$

Conclusion: The box needs to be $3' \times 6' \times 8'$

② Cost is the constraint

Ex3/ Same Set-Up as $\boxed{\text{Ex1}}$. But, if we have a budget of \$240, what is the largest box we can make?

Obj: Maximize $V_{(w,h)} = 2w^2h$

Given:

$$240 = 60hw + 80w^2$$

$$240 - 80w^2 = 60hw$$

$$h = \frac{240 - 80w^2}{60w} = \frac{12 - 4w^2}{3w}$$

$V(w) = 2w^2\left(\frac{12-4w^2}{3w}\right) = \frac{2}{3}w(12-4w^2) = 8w - \frac{8}{3}w^3$

$\frac{dV}{dw} = 8 - 8w^2 \overset{set}{=} 0$

$8w^2 = 8$

$w^2 = 1$

$w = \pm 1 \xrightarrow{w>0} \boxed{w = 1}$

Verify it is a max:

$\frac{d^2V}{dw^2} = -16w \Rightarrow \left.\frac{d^2V}{dw^2}\right]_{x=1} = -16 < 0$

Local Max + Only 1 Crit. Num $\Longrightarrow$ Abs. Max

Length: $2(1) = 2$

Height: $h\Big]_{w=1} = \frac{12-4\cdot 1}{3} = \frac{8}{3}$

Volume:

$V = 2w^2h$

$V(1,\frac{8}{3}) = 2\cdot 1^2 \cdot \frac{8}{3}$

$= \frac{16}{3} = 5 + \frac{1}{3}$

Conclusion: The largest box we can make has a volume of $(5+\frac{1}{3})\,ft^3$.

Revenue

Revenue = Total Money Earned

$= \underbrace{\left(\frac{\$\text{Price}}{\text{Unit}}\right)}_{p}\underbrace{(\#\text{ of Units})}_{q}$

Ex3/ A company's marketing department says that the number of units sold start at \$720 then decreases by 15 units for every \$1 increase in the price.

ⓐ What should the price be to maximize revenue?

$$\text{Obj: } \overset{\text{Max}}{R_{(p,q)}} = pq$$

$$\text{Constraint: } q = 720 - 15p$$

$$R_{(p)} = p(720 - 15p) = 720p - 15p^2$$

$$\frac{dR}{dp} = 720 - 30p \overset{\text{set}}{=} 0$$

$$p = \frac{720}{30} = \$24$$

Profit: Profit = Money earned after costs are dealt with

= Revenue − Cost

$$\text{Profit} = \left(\frac{\text{Price}}{\text{Unit}}\right)(\#\text{ of Units}) - \left(\frac{\text{Cost}}{\text{Unit}}\right)(\#\text{ of Units})$$

$$P_{(p,q)} = pq - \underbrace{c}_{\text{cost}}q$$

ⓑ What should the price be to maximize profit if it costs \$12 to make each item

$$\text{Obj: Max } P_{(p,q)} = pq - 12q$$

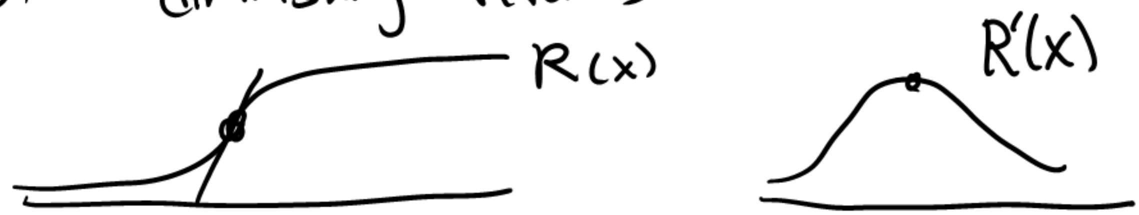
$$\text{Given: } q = 720 - 15p$$

$$P_{(p)} = p(720-15p) - 12(720-15p)$$
$$= 720p - 15p^2 - 8640 + 180p$$
$$P_{(p)} = -15p^2 + 900p - 8640$$
$$\frac{dP}{dp} = -30p + 900 \stackrel{\text{set}}{=} 0$$
$$p = \frac{900}{30} = \$30$$

Conclusion: The price needed to increase from \$24 to \$30 when factoring in costs.

Point of Diminishing Returns

When the derivative of (cost / Revenue / Profit) is at a maximum, this is called the point of diminishing returns



Ex 5/ A company uses the fcn
$$R(x) = 10x^2 - \frac{2}{3}x^3; \quad 0 \le x \le 10$$
to model revenue after spending x million dollars in advertising. Find and interpret the point of diminishing returns.

Obj: Maximize $\frac{dR}{dx} = 20x - 2x^2$

Given: $x \in [0, 10]$

$$\frac{d}{dx}\left(\frac{dR}{dx}\right) = 20 - 4x \underset{=}{\text{set}} 0$$

$$x = \frac{20}{4} = 5$$

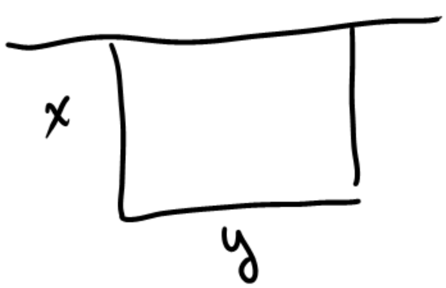
Verify it is a max:

$$\frac{d^2}{dx^2}\left(\frac{dR}{dx}\right) = -4 < 0$$

Likewise, plugging in $x = 0, 10$ into $\frac{dR}{dx}$ is 0. So it is abs. max.

Conclusion: The point of diminishing returns occurs at \$5 million. Ie, at \$5 million, the rate the revenue increases by spending more on advertising starts to slow.

Ex (HW 25, Q1)



If the fence cost \$55/ft, minimize costs given the area is 490 000 ft^2

Obj: Minimize $C(x,y) = 55(2x + y)$

Given: $490{,}000 = xy$