Goal: Solve optimization problems involving cost, revenue, and profit.

Costs can be incorporated in two ways

① As the objective fcn

Ex) Jessica wants to make a box such that

(i) The volume is $144 ft^3$

(ii) The length is double the width

(iii) The top and bottom are made out of metal

(iv) The sides are made out of wood.

If it costs $\$10/ft^2$ for wood and $\$20/ft^2$ for metal, what should the dimensions be to make this box as cheaply as possible?

$$\text{Cost} = \begin{pmatrix}\text{Cost of}\\ \text{Wood}\end{pmatrix} + \begin{pmatrix}\text{Cost of}\\ \text{Metal}\end{pmatrix}$$

$$\text{Cost} = \left(\frac{\text{Price of wood}}{\text{sq. ft.}}\right)(\#\text{ sq. ft}) + \left(\frac{\text{Price of Metal}}{\text{sq. ft}}\right)(\text{sq. ft.})$$

$$= 10\left(2wh + 2(2wh)\right) + 20\left(2(2w^2)\right)$$

$$C(w,h) = 60hw + 80w^2$$

Obj: Minimize $C(w,h) = 60hw + 80w^2$

Given: $144 = 2w^2 h$

$$h = \frac{144}{2w^2} = \frac{72}{w^2}$$

$$C(w) = 60\left(\frac{72}{w^2}\right)w + 80w^2 = \frac{4320}{w} + 80w^2$$

$$\frac{dC}{dw} = -\frac{4320}{w^2} + 160w \overset{\text{set}}{=} 0$$

$$160w = \frac{4320}{w^2} \quad ; \quad w > 0$$

$$160w^3 = 4320$$

$$w^3 = \frac{4320}{160} = 27$$

$$\boxed{w = 3}$$

Verify it's a minimum:

Local Min $\Longrightarrow$ Abs. Min
+
Only 1 crit. num.

Width: 3

Length: $2w \Rightarrow 2(3) = \boxed{6 \leftarrow \text{Length}}$

Height: $h = \frac{72}{w^2} \Rightarrow h\Big]_{w=3} = \frac{72}{3^2} = 8$

$$\begin{array}{r} 48 \\ \times 3 \\ \hline 144 \end{array}$$

<u>Conclusion:</u> The box needs to be $3' \times 6' \times 8'$ to minimize costs.

② Cost is a constraint

Ex2/ Same setup as $\boxed{\text{Ex1}}$. But, if we have a budget of \$240, what is the largest box we can make?

Obj: Maximize $V(w,h) = 2w^2h$

Given: $240 = 60hw + 80w^2$

$$240 - 80w^2 = 60hw$$

$$h = \frac{240 - 80w^2}{60w} = \frac{12 - 4w^2}{3w}$$

$$V(w) = 2w^2\left(\frac{12-4w^2}{3w}\right) = \frac{2}{3}w\left(12 - 4w^2\right)$$

$$V(w) = 8w - \frac{8}{3}w^3$$

$$\frac{dV}{dw} = 8 - 8w^2 \stackrel{\text{set}}{=} 0$$

$$8w^2 = 8$$

$$w^2 = 1$$

$$w = \pm 1 \xrightarrow{w>0} w = 1$$

Verify it's a max:

$$\frac{d^2V}{dw^2} = -16w \Rightarrow \left.\frac{d^2V}{dw^2}\right]_{w=1} = -16 < 0$$

Width: 1

Length: $2(1) = 2$

Height: $h\Big]_{w=1} = \frac{12 - 4(1)^2}{3(1)} = \frac{8}{3}$

$$V\left(1, \tfrac{8}{3}\right) = 2(1)^2\,\frac{8}{3} = \frac{16}{3}$$

$$= 5 + \frac{1}{3}$$

Conclusion: The largest box we can make has a volume of $\left(5 + \frac{1}{3}\right) ft^3$

Revenue

Revenue = Total Money Earned

$$R(p,q) = \underbrace{\left(\frac{\text{Price}}{\text{Unit}}\right)}_{p}\underbrace{\left(\#\text{ of Units}\right)}_{q}$$

Ex3/ A company's marketing department says that the number of units sold starts at 720, then decreases by 15 for every dollar increase in price.

(a) What should the price be to maximize revenue?

Obj: Maximize $R_{(p,q)} = pq$

Given: $q = 720 - 15p$

$$R_{(p)} = p(720 - 15p) = 720p - 15p^2$$

$$\frac{dR}{dp} = 720 - 30p \overset{\text{set}}{=} 0$$

$$p = \frac{720}{30} = \$24$$

<u>Profit</u>: Profit = The amount earned after costs are dealt with

= Revenue - Costs

$$= \left(\frac{\text{Price}}{\text{Unit}}\right)(\#\text{ units}) - \left(\frac{\text{Cost}}{\text{unit}}\right)(\#\text{ units})$$

$$P_{(p,q)} = pq - Cq$$

(b) What should the price be to maximize profit if it cost \$12 to make a unit?

Obj: Maximize $P_{(p,q)} = pq - 12q$

Given: $q = 720 - 15p$

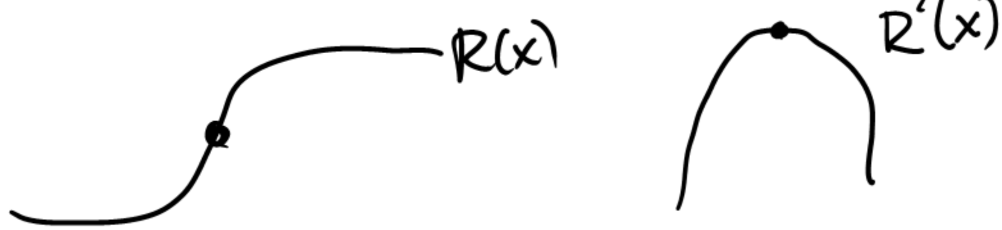
$$P_{(p)} = p(720-15p) - 12(720-15p)$$
$$= 720p - 15p^2 - 8640 + 180p$$
$$P_{(p)} = -15p^2 + 900p - 8640$$

$$\frac{dP}{dp} = -30p + 900 \ \underline{\underline{\text{set}}}\ 0$$
$$p = \frac{900}{30} = \$30$$

Conclusion: The price rose from \$24 to \$30 when factoring in costs.

Point of Diminishing Returns

When the derivative of (Cost/Revenue/Profit) is at a maximum, this is called the point of diminishing returns.



Ex5) A company uses the fcn
$$R(x) = 10x^2 - \frac{2}{3}x^3; \quad x \in [0, 10]$$
to model the revenue after spending x million dollars in advertising. Find and interpret the point of diminishing returns.

Obj: Maximize $\frac{dR}{dx} = 20x - 2x^2$

Given: $x \in [0, 10]$

$$\frac{d}{dx}\left(\frac{dR}{dx}\right) = 20 - 4x \overset{\text{set}}{=} 0$$

$$20 = 4x$$

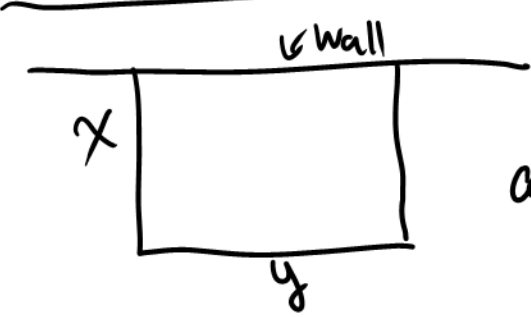
$$x = 5$$

Verify it is the abs. max.

x	0	5	10
$\frac{dR}{dx}$	0	$\underbrace{50}_{\text{max}}$	0

Conclusion: The point of diminishing returns is at \$5 million. I.e., at \$5 million, the rate that the revenue increases by spending more on advertising starts to slow.

Ex6 (HW 25, Q1)



If it costs \$55/ft of fence, what is the minimum cost to produce an enclosed space with area 490,000 ft^2

Obj: Min $C_{(x,y)} = 55(2x + y)$

Given: $490\,000 = xy$