Lecture 26: Antiderivatives and Indefinite Integration

Goal: Reverse the process of differentiation.

$\int kf(x)\ dx = kF(x) + C$	$\int (f(x) + g(x))\ dx = F(x) + G(x) + C$
$\int f(g(x))g'(x)\ dx = F(g(x)) + C$	$\int (f(x) - g(x))\ dx = F(x) - G(x) + C$
$\int x^n\ dx = \frac{1}{n+1}x^{n+1} + C$ (when $n \neq -1$)	$\int \frac{1}{x}\ dx = \ln\lvert x\rvert + C$
$\int e^x\ dx = e^x + C$	$\int e^{kx}\ dx = \frac{1}{k}e^{kx} + C$
$\int \sin x\ dx = -\cos x + C$	$\int \cos x\ dx = \sin x + C$
$\int \sec^2 x\ dx = \tan x + C$	$\int \sec x \tan x\ dx = \sec x + C$
$\int \csc x \cot x\ dx = -\csc x + C$	$\int \csc^2 x\ dx = -\cot x + C$
$\int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1} x + C$	$\int \frac{1}{1+x^2}\ dx = \tan^{-1} x + C$

Table 1: Table of Antiderivatives

Def A function F is an antiderivative of f if $F'(x) = f(x)$ for all x in I.

Ex/ (a) Is $F(x) = \frac{1}{2}x^2$ an antiderivative of x on $(-\infty, \infty)$.

$$F'(x) = \frac{1}{2} \cdot 2x = x \checkmark$$

(b) Is $F(x) = x^2$ an antiderivative of x

$$F'(x) = 2x \neq x \quad \text{No}$$

(c) Is $F(x) = \frac{1}{2}x^2 + 1$ an antiderivative for x

$$F'(x) = \left[\tfrac{1}{2}x^2\right]' + 1' = x + 0 = x \checkmark$$

NOTE: Antiderivatives are not unique $\underbrace{\frac{1}{2}x^2 + C}_{\text{another antiderivative}}$

Theorem If $F' = 0$ on an interval, then F is constant on the interval.

Why? Uses Mean Value Theorem

Corollary If F and G are antiderivatives of a function f, then

$$F(x) = G(x) + C$$

where C is a constant.

Why? $\frac{d}{dx}(F(x) - G(x)) = f(x) - f(x) = 0$

$$\Rightarrow F(x) - G(x) = [\text{Constant}]$$

Def The family of antiderivatives of a function f is called the (indefinite) integral of f, denoted

$$\underbrace{\int}_{\text{Integral Sign}} \overbrace{f(x)}^{\text{Integrand}} \, d\underbrace{x}_{\text{Integration Variable}} \overset{\text{By Corollary}}{=} \underbrace{F(x)}_{\text{Particular Antiderivative}} + \underbrace{C}_{\text{Constant of Integration}}$$

Ex 3/ Find all possible antiderivatives of $\sin x$

① Find a particular antiderivative

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$(-1)\frac{d}{dx}(\cos x) = \sin x$$

$$\frac{d}{dx}(-\cos x) = \sin x$$

② Compute $\boxed{\int \sin x \, dx = -\cos x + C}$

Ex3/ Compute $\int \frac{1}{x}\,dx$ on $(0,\infty)$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \ln x + C$$

In general, on $(-\infty, 0) \cup (0,\infty)$

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

Ex4/ Compute $\int x^2\,dx$

$$\frac{d}{dx}(x^3) = 3x^2$$

$$\frac{1}{3}\frac{d}{dx}(x^3) = x^2$$

$$\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$$

$$\int x^2\,dx = \frac{1}{3}x^3 + C$$

Theorem (Reverse Power Rule) When $n \neq -1$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

Ex5/ $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$

$$= \frac{2}{3}x^{\frac{3}{2}} + C$$

Theorem Indefinite Integrals are linear. I.e. if F and G are antiderivatives of f and g and k is a constant

① $$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx = F(x) \pm G(x) + C$$

② $$\int k f(x)\,dx = k\int f(x)\,dx = kF(x) + C$$

Ex 6 $$\int \left(\frac{2x^5 - \sqrt{x}}{x}\right)dx = \int\left(\frac{2x^5}{x} - \frac{\sqrt{x}}{x}\right)dx = \int (2x^4 - x^{-\frac{1}{2}})\,dx$$

$$= \underbrace{\int 2x^4\,dx} - \int x^{-\frac{1}{2}}\,dx$$

$$= 2\int x^4\,dx - \int x^{-\frac{1}{2}}\,dx$$

$$= 2\cdot\frac{x^{4+1}}{4+1} - \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{2}{5}x^5 - 2x^{\frac{1}{2}} + C$$

Remark $$\int (x+x^2)\,dx = \int x\,dx + \int x^2\,dx$$

$$= \frac{x^{1+1}}{1+1} + C_1 + \frac{x^{2+1}}{2+1} + C_2$$

$$= \frac{1}{2}x^2 + \frac{1}{3}x^3 + \underbrace{(C_1 + C_2)}_{=C}$$

Ex7/ $\int \sec x(\sec x + \cos x)\,dx = \int (\sec^2 x + \sec x \cos x)\,dx$

$\frac{d}{dx}[x] = 1$

$= \int \sec^2 x\,dx + \int \underbrace{\sec x}_{=\frac{1}{\cos x}} \cos x\,dx$

$= \int \sec^2 x\,dx + \int 1 \cdot x^0\,dx$

$= \tan x + x + C$

Remark Add parantheses when the integrand has more than 1 terms

$\int x^2 + x^3\,dx$ ✗

$\int (x^2 + x^3)\,dx$ ✓

This won't be on the test, but you'll see it in MA1602D

Theorem (Reverse Chain Rule) If F is an antiderivative for f. Then,

$$\int f(g(x)) \cdot g'(x)\,dx = F(g(x)) + C$$

Ex8/ $\int \cos(2x)\,dx$

$\frac{d}{dx}(\sin(2x)) = 2 \cdot \cos(2x)$ Want this in our integrand

$\frac{2}{2}\int \cos(2x)\,dx = \frac{1}{2}\int 2\cos(2x)\,dx = \frac{1}{2}\sin(2x) + C$