Goal: Be able to solve (very basic) Initial Value Problems through integration.

For $y=f(x)$, a <u>differential equation</u> (diff.eq) is an equation involving x, y, and the derivatives of y.

E.g.,

$$\frac{d^2y}{dx^2} = -9.81 \quad [\text{Gravity}]$$

$$\frac{dy}{dx} = y \quad [\text{Exponential Growth}]$$

$$\frac{d^2y}{dx^2} + y = \cos x \quad [\text{Dampened Harmonic Motion}]$$

Ex) Verify that $y = 3e^{2x}$ is a sol. to $\frac{dy}{dx} = 2y$

$$\frac{d}{dx}(3e^{2x}) = 2 \cdot 3e^{2x} = 6e^{2x}$$

$$2(3e^{2x}) = 6e^{2x}$$

Ex2) Solve the diff. eq. $y' = \frac{1}{x}$

$$y' = \frac{1}{x}$$

$$\int y'\,dx = \int \frac{1}{x}\,dx$$

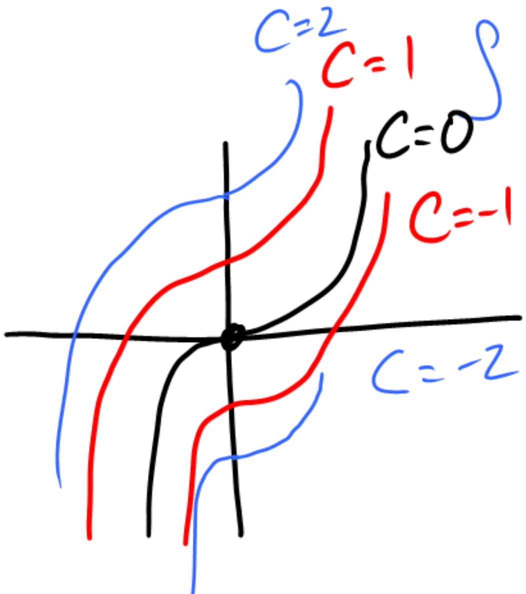
$$y = \ln|x| + C$$

General Sol. to $y' = \frac{1}{x}$

A function describing all possible solutions is called the <u>general solution</u> to the diff. eq.

Ex 3/ Find the general solution to

$$y' = 3x^2$$



$$\int y' \, dx = \int 3x^2 \, dx$$

$$y = 3 \int x^2 \, dx$$

$$y = 3 \cdot \frac{x^{2+1}}{2+1} + C$$

$$y = x^3 + C$$

Q: Is there a way to get unique (only 1) sol?

A: Say y also needs through a point (e.g., $y(0) = 0$). Ie, specify an initial condition.

✱ Usually we specify when $x=0$, but that's not required.

IVPs

An initial value problem (IVP) is a diff eq. combined with initial conditions

Ex 5/ The system

$$\begin{cases} y' = \sqrt{x} \\ y(0) = 1 \end{cases}$$

is an IVP. How can we solve this?

① Find the general solution

$$y' = \sqrt{x}$$

$$\int y'\,dx = \int \sqrt{x}\,dx$$

$$y = \int x^{\frac{1}{2}}\,dx$$

$$y = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$y = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C \Rightarrow y = \frac{2}{3}x^{\frac{3}{2}} + C$$

(2) Plug in IC, solve for C $\boxed{y(0) = 1}$

$$1 = \frac{2}{3}(0)^{\frac{3}{2}} + C$$

$$\boxed{1 = C}$$

The solution to the IVP is $y = \frac{2}{3}x^{\frac{3}{2}} + 1$

Def y is <u>proportional to</u> x if there is a constant k such that $y = kx$

Ex) The size of the population is governed by

$$\frac{dP}{dt} = \sqrt[3]{t}$$

$P(t)$ = Population size after t days

(ie, $\frac{dP}{dt}$ is proportional to $\sqrt[3]{t}$ where $k=1$). If the initial population is 1000, find the pop. size after 8 days

Ie, we need to solve IVP

$$\begin{cases} \frac{dP}{dt} = \sqrt[3]{t} \\ P(0) = 1000 \end{cases}$$

$$\frac{dP}{dt} = \sqrt[3]{t} \qquad t^{\frac{1}{3}}$$

$$\int \frac{dP}{dt}\, dt = \int \sqrt[3]{t}\, dt$$

$$P_{(t)} = \frac{t^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C$$

$$P(t) = \frac{3}{4} \cdot t^{\frac{4}{3}} + C$$

(2) Plug in IC

$$1000 = P(0) = \frac{3}{4}(0)^{\frac{4}{3}} + C$$

$$C = 1000$$

Sol: $P(t) = \frac{3}{4} t^{\frac{4}{3}} + 1000$

$$P(8) = \frac{3}{4} \cdot 8^{\frac{4}{3}} + 1000 = \frac{3}{4}\left(\sqrt[3]{8}\right)^4 + 1000$$

$$= 12 + 1000 = 1012$$

Conclusion: The population is 1012 after 8 days

Multiple ICs

Fact: To have a unique sol., we need N ICs where N is the highest order of derivatives in the diff eq

Ex 7 Solve the IVP

$$\begin{cases} y''(t) = t-3 \\ y'(1) = \frac{1}{2} \\ y(0) = 1 \end{cases}$$

$$\begin{tikzcd} y \arrow[r, "d/dx"] & y' \arrow[r, "d/dx"] \arrow[l, bend left, "\int"] & y'' \arrow[l, bend left, "\int"] \end{tikzcd}$$

$$y''(t) = t-3$$

$$\int y''(t)\,dt = \int (t-3)\,dt$$

$$y'(t) = \frac{1}{2}t^2 - 3t + C$$

$y'(1) = \frac{1}{2}$

$$\frac{1}{2} = y'(1) = \frac{1}{2}(1)^2 - 3(1) + C$$

$$\frac{1}{2} = \frac{1}{2} - 3 + C$$

$$0 = -3 + C$$

$$C = 3$$

Thus,

$$y'(t) = \frac{1}{2}t^2 - 3t + 3$$

$$\int (y'(t))\,dt = \int \left(\frac{1}{2}t^2 - 3t + 3\right)dt$$

$$y(t) = \frac{1}{2}\cdot\frac{t^{2+1}}{2+1} - 3\cdot\frac{t^{1+1}}{1+1} + 3t + D$$

$$y(t) = \frac{1}{6}t^3 - \frac{3}{2}t^2 + 3t + D$$

$y(0) = 1$ $\quad 1 = y(0) = 0 + D \Rightarrow D = 1$

Conclusion: $y(t) = \frac{1}{6}t^3 - \frac{3}{2}t^2 + 3t + 1$

NOTE: You can always check your solution by differentiating

$y'(t) = \frac{1}{2}t^2 - 3t + 3$

$y'(1) = \frac{1}{2} - 3 + 3 = \frac{1}{2}$ ✓

$y''(t) = t - 3$ ✓

$y(0) = 1$ ✓

Ex 8/ A ball is kicked from the ground with an initial velocity of $20\frac{m}{s}$. Describe the ball's vertical height as a fcn of time (assume gravity is the only force acting on the ball)

Ie, solve the system

$$\begin{cases} a(t) = -9.81 & \leftarrow s'' \\ v(0) = 20 & \leftarrow s' \\ s(0) = 0 \end{cases}$$

accel $\xrightarrow{\int}$ velocity $\xrightarrow{\int}$ pos.

$a(t) = -9.81$

$\int a(t)\,dt = \int(-9.81)\,dt \leftarrow (-9.81)\int 1\,dt$

$$v(t) = -9.81t + C$$

$v(0) = 20$

$$20 = v(0) = -9.81(0) + C$$

$$C = 20$$

Ie,

$$v(t) = -9.81t + 20$$

$$\int v(t)\,dt = \int (-9.81t + 20)t$$

$$s(t) = (-9.81)\cdot\frac{1}{2}t^2 + 20t + D$$

$$s(t) = -4.905t^2 + 20t + D$$

$s(0) = 0$

$$0 = s(0) = 0 + D$$

$$D = 0$$

Ie,

$$s(t) = -4.905t^2 + 20t$$

<u>Ex 9 (HW 27, Q4)</u> Solve the IVP

$$\begin{cases} y' = 3\cos x + 5 \\ y\left(\frac{3\pi}{2}\right) = 4 \end{cases}$$

Sol/

$$y' = 3\cos x + 5$$

$$\int y'\,dx = \int (3\cos x + 5)\,dx$$

$$y = 3\sin x + 5x + C$$

$y\left(\frac{3\pi}{2}\right) = 4$

$$4 = y\left(\frac{3\pi}{2}\right) = 3\sin\left(\frac{3\pi}{2}\right) + 5\left(\frac{3\pi}{2}\right) + C$$

$$4 = 3(-1) + \frac{15\pi}{2} + C$$

$$4 = -3 + \frac{15\pi}{2} + C$$

$$\boxed{7 - \frac{15\pi}{2} = C}$$