Goal: Be able to solve (very basic) Initial Value Problems through integration.

For $y=f(x)$, a <u>differential equation</u> (diff.eq) is an equation involving x, y, and the derivatives of y.

E.g.,

$$\frac{d^2y}{dx^2} = -9.81 \qquad \text{[Gravity]}$$

$$\frac{dy}{dx} = y \qquad \text{[Exponential Growth]}$$

$$\frac{d^2y}{dx^2} + y = \cos x \qquad \text{[Dampened Harmonic Motion]}$$

Ex1/ Verify $y=3e^{2x}$ is a solution to the diff eq.

$$\frac{dy}{dx} = 2y$$

$$\frac{d}{dx}(3e^{2x}) = 2\cdot 3e^{2x} = 6e^{2x}$$

$$2(3e^{2x}) = 6e^{2x}$$

Ex2/ Solve the diff. eq. $y' = \frac{1}{x}$

$$y' = \frac{1}{x}$$

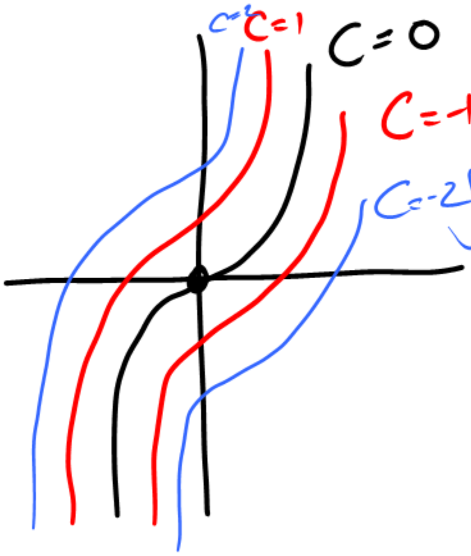
$$\int y'\,dx = \int \frac{1}{x}\,dx$$

$$y = \ln|x| + C$$

General Sol. to $y' = \frac{1}{x}$

A function describing all possible solutions is called the <u>general solution</u> to the diff eq

Ex3/ Find the general sol. to $y' = 3x^2$



$$y' = 3x^2$$

$$\int y' \, dx = \int 3x^2 \, dx$$

$$y(x) = 3 \int x^2 \, dx$$

$$y(x) = 3 \cdot \frac{x^{2+1}}{2+1} + C$$

$$y(x) = x^3 + C$$

Q: Is there a way to have a unique (only 1) solution?

A: Say y also needs to pass through a point (e.g. $y(0) = 0$). This is called an initial condition.

☆ Usually we specify when $x=0$, but that's not always.

IVPs

An initial value problem (IVP) is a diff eq combined with initial conditions

Ex 5/ The system

$$\begin{cases} y'(x) = \sqrt{x} \\ y(0) = 1 \end{cases}$$

is an IVP. How can we solve?

① Find the general solution

$$y'(x) = \sqrt{x}$$

$$\int y'(x)\,dx = \int \sqrt{x}\,dx$$

$$y(x) = \int x^{\frac{1}{2}}\,dx$$

$$y(x) = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$y(x) = \frac{2}{3}x^{\frac{3}{2}} + C$$

② Plug in IC, Solve for C $y(0)=1$

$$1 = y(0) = \frac{2}{3}(0)^{\frac{3}{2}} + C$$

$$1 = 0 + C$$

$$C = 1$$

Conclusion: $y(x) = \frac{2}{3}x^{\frac{3}{2}} + 1$ is the sol. to the IVP.

Def We say y is proportional to x if there is a constant k s.t. $y = kx$

Ex6/ The size of a population is governed by

$$\frac{dP}{dt} = \sqrt[3]{t} \leftarrow \frac{dP}{dt} \text{ is proportional to } \sqrt[3]{t} \text{ where } k=1$$

where P is the population after t days. If the initial population is 1000, find the pop. size after 8 days.

Ie, solve

$$\begin{cases} \frac{dP}{dt} = \sqrt[3]{t} \\ P(0) = 1000 \end{cases}$$

$$\frac{dP}{dt} = \sqrt[3]{t}$$

$$\int \frac{dP}{dt}\, dt = \int \sqrt[3]{t}\, dt$$

$$P(t) = \int t^{\frac{1}{3}}\, dt$$

$$P(t) = \frac{t^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C$$

$P(0) = 1000$

$$P(t) = \frac{3}{4}\underbrace{t^{\frac{4}{3}}}_{0} + C$$

$$1000 = P(0) = 0 + C$$

$$C = 1000$$

$$P(t) = \frac{3}{4} t^{\frac{4}{3}} + 1000$$

Thus,

$$P(8) = \frac{3}{4}\, 8^{\frac{4}{3}} + 1000 = \frac{3}{4}\left(\sqrt[3]{8}\right)^4 + 1000$$

$$= 12 + 1000 = 1012$$

Conclusion: The population size is 1,012 after 8 days.

Multiple ICs

Fact: To have a unique solution, you need N ICs where N is the highest order of derivatives in the diff eq.

Ex7/ Solve the IVP

$$\begin{cases} y''(t) = t-3 \\ y'(1) = \frac{1}{2} \\ y(0) = 1 \end{cases}$$

$$\begin{tikzcd} y \arrow[r, "d/dt"] & y' \arrow[r, "d/dt"] \arrow[l, bend left, "\int"] & y'' \arrow[l, bend left, "\int"] \end{tikzcd}$$

$$y''(t) = t-3$$

$$\int y''(t)\,dt = \int (t-3)\,dt$$

$$y'(t) = \frac{t^{1+1}}{1+1} - 3t + C$$

$$y'(t) = \frac{1}{2}t^2 - 3t + C$$

$\underline{y'(1) = \frac{1}{2}}$

$$\frac{1}{2} = y'(1) = \frac{1}{2} - 3 + C$$

$$\frac{1}{2} = \frac{1}{2} - 3 + C$$

$$0 = -3 + C$$

$$\rightarrow C = 3$$

$$y'(t) = \frac{1}{2}t^2 - 3t + 3$$

$$\int y'(t)\,dt = \int \left(\frac{1}{2}t^2 - 3t + 3\right) dt$$

$$y(t) = \frac{1}{2}\cdot\frac{t^{2+1}}{2+1} - 3\cdot\frac{t^{1+1}}{1+1} + 3t + D$$

$$y(t) = \frac{1}{6}t^3 - \frac{3}{2}t^2 + 3t + D$$

$\underline{y(0) = 1}$

$$1 = y(0) = 0 + D$$

$$D = 1$$

Thus, the solution is $y(t) = \frac{1}{6}t^3 - \frac{3}{2}t^2 + 3t + 1$

NOTE: You can always check your answer by differentiating

$$y(0) = 0 - 0 + 0 + 1 = 1 \checkmark$$

$$y'(t) = \frac{1}{2}t^2 - 3t + 3$$

$$y'(1) = \frac{1}{2} - 3 + 3 = \frac{1}{2} \checkmark$$

$$y''(t) = t - 3 \checkmark$$

Ex 8/ A ball is kicked from the ground with an initial velocity of $20\frac{m}{s}$. Describe the vertical height of the ball as a fcn of time (assume gravity is the only force acting on the ball)

Solve the system

$$\begin{cases} s'' \rightarrow a(t) = -9.81 \\ s' \rightarrow v(0) = 20 \\ s \rightarrow s(0) = 0 \end{cases}$$

$$\text{Accel.} \xrightarrow{\int} \text{velocity} \xrightarrow{\int} \text{pos.}$$

$$a(t) = -9.81$$

$$\int a(t)\,dt = \int (-9.81)\,dt$$

$$v(t) = -9.81t + C$$

$v(0) = 20$

$$20 = v(0) = -9.81(0) + C$$

$$C = 20$$

$$v(t) = -9.81t + 20$$

$$\int v(t)\,dt = \int (-9.81t + 20)\,dt$$

$$s(t) = -9.81\left(\tfrac{1}{2}t^2\right) + 20t + D$$

$$s(t) = -4.905t^2 + 20t + D$$

$s(0) = 0$

$$0 = s(0) = D$$

$$D = 0$$

Thus, $s(t) = -4.905t^2 + 20t$

Ex 9 (HW27, Q4) Solve the IVP

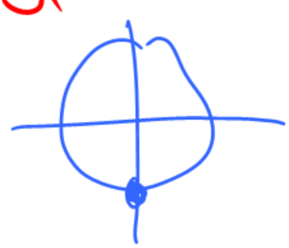
$$\begin{cases} y' = 3\cos x + 5 \\ y\left(\frac{3\pi}{2}\right) = 4 \end{cases}$$

$$y' = 3\cos x + 5$$

$$\int y'\,dx = \int (3\cos x + 5)\,dx$$

$$y(x) = 3\sin x + 5x + C$$

$y\left(\frac{3\pi}{2}\right) = 4$



$$4 = y\left(\tfrac{3\pi}{2}\right) = 3\sin\left(\tfrac{3\pi}{2}\right) + 5\left(\tfrac{3\pi}{2}\right) + C$$

$$4 = 3(-1) + \tfrac{15\pi}{2} + C$$

$$4 = -3 + \tfrac{15}{2}\pi + C$$

$$7 = \tfrac{15\pi}{2} + C$$

$$C = 7 - \tfrac{15}{2}\pi$$

Thus, $y(x) = 3\sin x + 5x + 7 - \frac{15}{2}\pi$