Goal: Understand the notation $\sum_{i=M}^{N} a_i$. Approximate the (signed) area underneath the curve.

Sigma Notation:

We want short hand for sums with a predictable pattern. We use the Greek letter Sigma (Σ) to do so.

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

n } Where to stop

i^{th} term

Index "Variable" → $i = m$

Where to start

Ex1/ $\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$

$\sum_{j=0}^{5} 2^j = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$

Ex2/ Express $2^3 + 3^3 + 4^3 + \cdots + n^3$ in Sigma notation

$$\sum_{i=2}^{n} i^3$$

NOTE (Reindexing): We can start from any number if we adjust our formula accordingly

$$\sum_{i=2}^{n} i^3 = \sum_{i=1}^{n-1} (i+1)^3 = \sum_{i=0}^{n-2} (i+2)^3$$

Ex3/ Express $(1)^2 + (2)^2 + (\sqrt{2}+1)^2 + (\sqrt{3}+1)^2 + \cdots + (\sqrt{n}+1)^2$

$(\sqrt{0}+1)^2$ $\quad$ $(\sqrt{1}+1)^2$

$$\sum_{i=0}^{n} (\sqrt{i}+1)^2$$

Theorem Sums are linear. Ie. If k is a constant

① $$\sum_{i=M}^{N} (a_i \pm b_i) = \sum_{i=M}^{N} a_i \pm \sum_{i=M}^{N} b_i$$

② $$\sum_{i=M}^{N} k a_i = k \sum_{i=M}^{N} a_i$$

Ex4/ Compute $\sum_{i=1}^{n} 1$

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ times}} = n$$

Ex5/ Show $\sum_{i=1}^{n} i = 1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$

$$S = 1 + 2 + 3 + \cdots + n$$

$$+ \quad S = n + (n-1) + (n-2) + \cdots + 1$$

$$2S = \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1)}_{n \text{ times}}$$

$$2S = n(n+1)$$

$$S = \frac{n(n+1)}{2}$$

Theorem ① $\sum_{i=1}^{n} n$

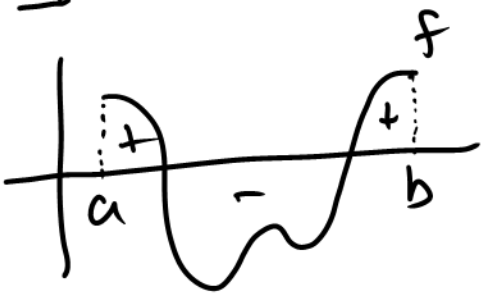
② $\sum_{i=1}^{n} k = kn$

③ $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

④ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

⑤ $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

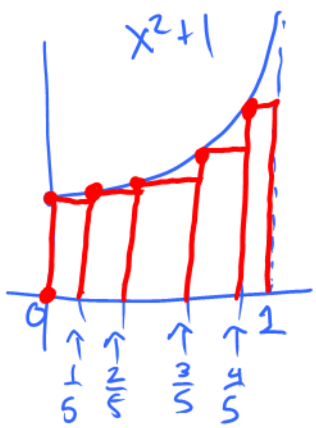
The Area Problem



Q: Given a function f and a closed interval $[a,b]$, find the (signed) area underneath the graph

A: Approximate the shape with rectangles

Ex/ Approximate the area of the region bounded $f(x) = x^2+1$, $x=0$, $x=1$, and the x-axis using 5 rectangles.



Width: $\frac{1}{5}$

$$\text{Area} \approx \frac{1}{5}f(0) + \frac{1}{5}f\left(\frac{1}{5}\right) + \frac{1}{5}f\left(\frac{2}{5}\right) + \frac{1}{5}f\left(\frac{3}{5}\right) + \frac{1}{5}f\left(\frac{4}{5}\right)$$

$$\text{Area} \approx \sum_{i=0}^{4} \frac{1}{5} f\left(\frac{i}{5}\right) = \sum_{i=0}^{4} \frac{1}{5}\left[\left(\frac{i}{5}\right)^2+1\right]$$

$$= \frac{1}{5}\sum_{i=0}^{4}\left[\left(\frac{i}{5}\right)^2+1\right]$$

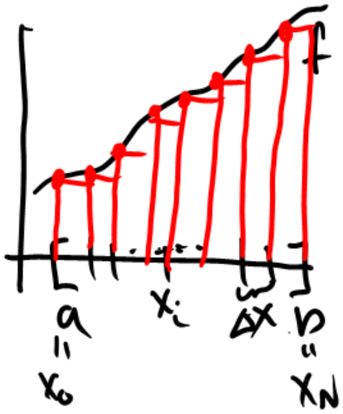
$$= \frac{1}{5}\left[\left(\frac{0}{5}\right)^2+1 + \left(\frac{1}{5}\right)^2+1 + \left(\frac{2}{5}\right)^2+1 + \left(\frac{3}{5}\right)^2+1 + \left(\frac{4}{5}\right)^2+1\right]$$

$$= \frac{1}{5}[5 + 0.04 + 0.16 + 0.36 + 0.64]$$

$$= \frac{1}{5}[6.2] = 1.24$$

This is an example of a <u>Left Riemann Sum</u> ("Left" is referring to how we chose the heights)

How do we do this in general



① Divide $[a,b]$ into N equally spaced segments

Width: $\dfrac{b-a}{N} = \Delta x$

② Determine Heights:

$$x_i = a + i\Delta x$$

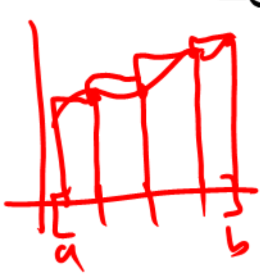
$$\text{Height}: f(x_i) = f(a+i\Delta x)$$

$$0 \le i \le N-1$$

<u>Def</u> The <u>Left Riemann Sum</u> of f over $[a,b]$ using N rectangles is defined as

$$L_N \overset{\text{def}}{=} \sum_{i=0}^{N-1} f(a+i\Delta x)\Delta x \quad \text{where } \Delta x = \frac{b-a}{N}$$

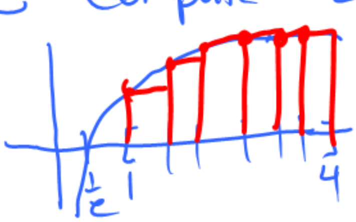
The <u>Right Riemann Sum</u> is



$$R_N \overset{\text{def}}{=} \sum_{i=1}^{N} f(a+i\Delta x)\Delta x \;;\; \Delta x = \frac{b-a}{N}$$

Ex 7/ Consider the function $f(x) = \ln(x)+1$ on $[1,4]$

ⓐ Compute L_6 [ie, the Left Riemann Sum w/ 6 Rectangles]



$$\Delta x = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$$

$$L_6 = \sum_{i=0}^{5} \left(\tfrac{1}{2}\right) f\left(1+i\left(\tfrac{1}{2}\right)\right) = \frac{1}{2}\sum_{i=0}^{5}\left[\ln\left(1+\frac{i}{2}\right)+1\right]$$

$$= \frac{1}{2}\sum_{i=0}^{5} 1 + \frac{1}{2}\sum_{i=0}^{5} \ln\left(1+\frac{i}{2}\right)$$

$$\star = \frac{1}{2}(6) + \frac{1}{2}\left[\ln(1)+\ln\left(\tfrac{3}{2}\right)+\ln(2)+\ln\left(\tfrac{5}{2}\right)+\ln(3)+\ln\left(\tfrac{7}{2}\right)\right]$$

$$= 3 + \frac{1}{2}\ln\left(1\cdot\frac{3}{2}\cdot 2\cdot\frac{5}{2}\cdot 3\cdot\frac{7}{2}\right) = 3+\frac{1}{2}\ln\left(\frac{630}{8}\right)$$

$$\approx 3 + \frac{1}{2}(4.366278) = 3 + 2.18314 \approx \boxed{5.183139}$$

(b) Repeat for R_6

$$\Delta x = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$$

$$R_6 = \sum_{i=1}^{6}\left(\tfrac{1}{2}\right) f\left(1+\frac{i}{2}\right) = \frac{1}{2}\sum_{i=1}^{6}\left[\ln\left(1+\frac{i}{2}\right)+1\right]$$

$$= \frac{1}{2}\sum_{i=1}^{6} 1 + \frac{1}{2}\sum_{i=1}^{6}\ln\left(1+\frac{i}{2}\right)$$

$$= 3 + \frac{1}{2}\left[\ln\left(\tfrac{3}{2}\right)+\ln(2)+\ln\left(\tfrac{5}{2}\right)+\ln(3)+\ln\left(\tfrac{7}{2}\right)+\ln(4)\right]$$

$$= 3 + \frac{1}{2}\ln\left(\frac{2520}{8}\right)$$

$$\approx \boxed{5.876286}$$

(c) Find (but do not compute) L_N and R_N

$$\Delta x = \frac{4-1}{N} = \frac{3}{N} \qquad a=1,\ b=4$$

$$L_N = \sum_{i=0}^{N-1} f(a + i\Delta x)\Delta x = \sum_{i=0}^{N-1}\left[\ln\left(1+\frac{3i}{N}\right)+1\right]\left(\frac{3}{N}\right)$$

$$= \frac{3}{N}\sum_{i=0}^{N-1}\left[\ln\left(1+\frac{3i}{N}\right)+1\right]$$

$$R_N = \frac{3}{N}\sum_{i=1}^{N}\left[\ln\left(1+\frac{3i}{N}\right)+1\right]$$