Goal: Understand the notation $\sum\limits_{i=M}^{N} a_i$. Approximate the (signed) area underneath the curve.

Sigma Notation:

We want short hand for sums with a predictable pattern. We use the Greek letter Sigma (Σ) to do so.

$$\sum_{i=m}^{n} a_i \overset{\text{def}}{=} a_m + a_{m+1} + \dots + a_n$$

n ← where to stop

"Variable" (index) → $i = m$ ← where to start

a_i: ith term

Ex1/ $\sum\limits_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$

$\sum\limits_{j=0}^{5} 2^j = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$

Ex2/ Express $2^3 + 3^3 + 4^3 + \dots + n^3$ in Sigma Notation

$$\sum_{i=2}^{n} i^3$$

NOTE (Reindexing) You can start from any number if you adjust your formula accordingly.

$$\sum_{i=2}^{n} i^3 = \sum_{i=1}^{n-1} (i+1)^3 = \sum_{i=0}^{n-2} (i+2)^3$$

Ex3/ $1^2 + 2^2 + (\sqrt{2}+1)^2 + (\sqrt{3}+1)^2 + \dots + (\sqrt{n}+1)^2$

1^2 ↑ $(\sqrt{0}+1)^2$ 2^2 ↑ $(\sqrt{1}+1)^2$

$$\sum_{i=0}^{n} (\sqrt{i}+1)^2$$

<u>Theorem</u> Sums are linear. Ie, if k is a constant (ie, not a fcn of the index)

① $$\sum_{i=m}^{n} (a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i$$

② $$\sum_{i=m}^{n} k a_i = k \sum_{i=m}^{n} a_i$$

Ex 4/ Compute $\sum_{i=1}^{n} 1$

$$\sum_{i=1}^{n} 1 = \underbrace{1+1+1+\cdots+1}_{n \text{ times}} = n$$

Likewise, for a constant k

$$\sum_{i=1}^{n} k = k \sum_{i=1}^{n} 1 = kn$$

Ex 5/ Show $\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

$$S = 1+2+3+\cdots+n$$

$$+\ S = n+(n-1)+(n-2)+\cdots+1$$

$$2S = \underbrace{(n+1)+(n+1)+(n+1)+\cdots+(n+1)}_{n \text{ times}}$$

$$2S = n(n+1)$$
$$S = \frac{n(n+1)}{2}$$

Theorem

① $\sum_{i=1}^{n} 1 = n$

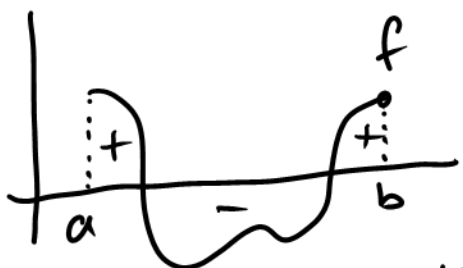
② $\sum_{i=1}^{n} k = kn$

③ $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

④ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

⑤ $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

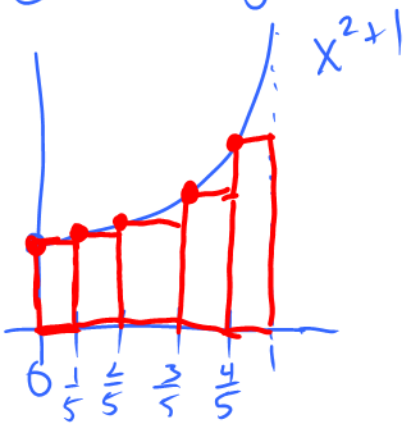
The Area Problem



Q: Given a fcn f and a closed interval $[a,b]$, compute the (signed) area underneath the curve.

A: Approximate the region via rectangles

Ex/ Approximate the area of the region bounded by $f(x) = x^2+1$, $x=0$, $x=1$, and the x-axis using 5 rectangles.



Width: $\frac{1}{5}$

$$\text{Area} \approx \frac{1}{5}f(0) + \frac{1}{5}f\left(\frac{1}{5}\right) + \frac{1}{5}f\left(\frac{2}{5}\right) + \frac{1}{5}f\left(\frac{3}{5}\right) + \frac{1}{5}f\left(\frac{4}{5}\right)$$

$$= \sum_{i=0}^{4} \frac{1}{5}f\left(\frac{i}{5}\right) = \sum_{i=0}^{4} \frac{1}{5}\left[\left(\frac{i}{5}\right)^2 + 1\right]$$

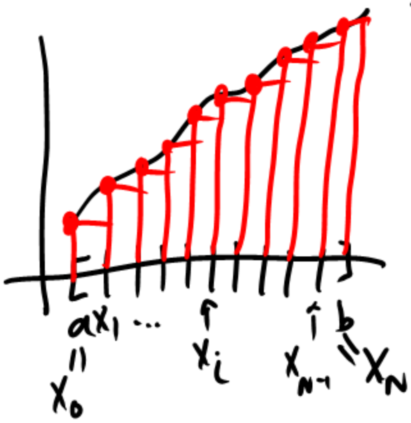
$$= \frac{1}{5}\left[\left(\frac{0}{5}\right)^2+1 + \left(\frac{1}{5}\right)^2+1 + \left(\frac{2}{5}\right)^2+1 + \left(\frac{3}{5}\right)^2+1 + \left(\frac{4}{5}\right)^2+1\right]$$

$= \frac{1}{5}[5 + 1.2] = \frac{1}{5}[6.2] = 1.24$

This is an example of a <u>Left Riemann Sum</u>

("Left" is referring to how we chose the height)

How do we do this in general?



① Divide $[a,b]$ into N equal region

Width: $\frac{b-a}{N} = \Delta x$

② Determine Heights

$x_i = a + i\Delta x$

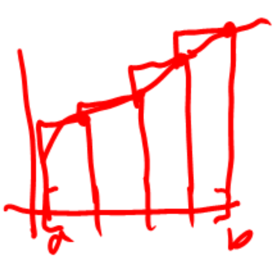
Heights: $f(x_i) = f(a+i\Delta x)$; $0 \le i \le N-1$

<u>Def</u> The <u>Left Riemann Sum</u> of f over $[a,b]$ using N rectangles is defined as

$$L_N \overset{def}{=} \sum_{i=0}^{N-1} f(a+i\Delta x)\Delta x \text{ where } \Delta x = \frac{b-a}{N}$$

Similiarly, the Right Riemann Sum is

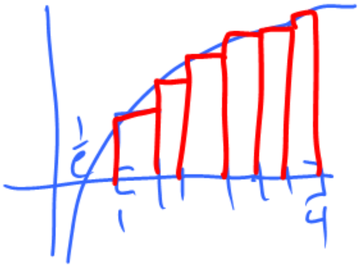
$$R_N \overset{def}{=} \sum_{i=1}^{N} f(a+i\Delta x)\Delta x \text{ where } \Delta x = \frac{b-a}{N}$$



Ex 7/ Consider $f(x) = \ln(x) + 1$ on $[1,4]$

ⓐ Compute L_6 [ie, left Riemann Sum w/ 6 rect.]

$$\Delta x = \frac{b-a}{N} \xrightarrow[N=6]{a=1,\ b=4} \Delta x = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$$



$$L_N = \sum_{i=0}^{N-1} f(a+i\Delta x)\Delta x$$

$$L_6 = \sum_{i=0}^{5} f\left(1+i\left(\tfrac{1}{2}\right)\right)\tfrac{1}{2} = \tfrac{1}{2}\sum_{i=0}^{5}\left[\ln\left(1+\tfrac{i}{2}\right)+1\right]$$

$$= \tfrac{1}{2}\sum_{i=0}^{5} 1 + \tfrac{1}{2}\sum_{i=0}^{5}\ln\left(1+\tfrac{i}{2}\right)$$

$$= \tfrac{1}{2}(6) + \tfrac{1}{2}\sum_{i=0}^{5}\ln\left(1+\tfrac{i}{2}\right)$$

$$= 3 + \tfrac{1}{2}\left[\ln(1)+\ln\left(\tfrac{3}{2}\right)+\ln(2)+\ln\left(\tfrac{5}{2}\right)+\ln(3)+\ln\left(\tfrac{7}{2}\right)\right]$$

$$= 3 + \tfrac{1}{2}\ln\left(1\cdot\tfrac{3}{2}\cdot 2\cdot\tfrac{5}{2}\cdot 3\cdot\tfrac{7}{2}\right) = 3+\tfrac{1}{2}\ln\left(\tfrac{630}{8}\right)$$

$$\approx 3+\tfrac{1}{2}(4.366278) = 3+2.18314 \approx \boxed{5.183139}$$

ⓑ Repeat for R_6

$$\Delta x = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$$

$$R_6 = \sum_{i=1}^{6}\tfrac{1}{2}\left[\ln\left(1+\tfrac{i}{2}\right)+1\right] = \tfrac{1}{2}\sum_{i=1}^{6}\left[\ln\left(1+\tfrac{i}{2}\right)+1\right]$$

$$= 3+\tfrac{1}{2}\sum_{i=1}^{6}\ln\left(1+\tfrac{i}{2}\right)$$

$= 3 + \frac{1}{2}\left[\ln\left(\frac{3}{2}\right) + \ln(2) + \ln\left(\frac{5}{2}\right) + \ln(3) + \ln\left(\frac{7}{2}\right) + \ln(4)\right]$

$= 3 + \frac{1}{2}\ln\left(\frac{3}{2} \cdot 2 \cdot \frac{5}{2} \cdot 3 \cdot \frac{7}{2} \cdot 4\right)$

$= 3 + \frac{1}{2}\ln\left(\frac{2520}{8}\right) \approx 3 + \frac{1}{2}(5.752573)$

$= 3 + 2.876286 = 5.876286$

ⓒ Set-Up (but do not compute) L_N and R_N

$\Delta x = \frac{b-a}{N} \Rightarrow \Delta x = \frac{4-1}{N} = \frac{3}{N}$

$$L_N = \sum_{i=0}^{N-1} f\left(1 + i\left(\frac{3}{N}\right)\right)\left(\frac{3}{N}\right)$$

$$L_N = \frac{3}{N}\sum_{i=0}^{N-1}\left[\ln\left(1 + \frac{3i}{N}\right) + 1\right]$$

$$R_N = \frac{3}{N}\sum_{i=1}^{N}\left[\ln\left(1 + \frac{3i}{N}\right) + 1\right]$$