**GOAL:** Interpret the definite integral as the (signed) area underneath the graph of a function.
Link for Desmos Presentation: here

Recall Left/Right Riemann Sums ($L_N$ and $R_N$) are used to approximate the area underneath the graph of a fcn.

Q: What happens when we take $N \to \infty$?

$$\lim_{N\to\infty} L_N = \lim_{N\to\infty} R_N = \text{The exact area}$$

Def The (definite) integral of a function f from a to b is

$$\int_a^b f(x)\,dx \overset{\text{def}}{=} \text{The signed area between the graph of } f \text{ and the } x\text{-axis on } [a,b].$$

b ← Upper Limit

a ← Lower Limit

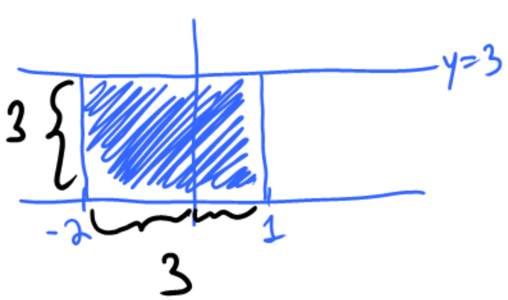
$$= \lim_{N\to\infty} L_N = \lim_{N\to\infty} R_N$$

a and b are called the limits of integration.

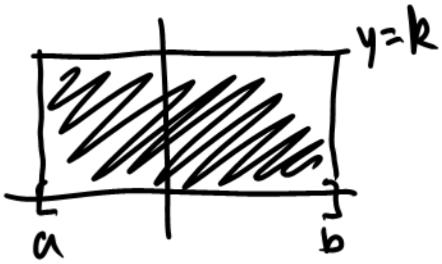
If $\lim_{N\to\infty} L_N$ exists, we say f is Riemann integrable on [a,b].

Ex/ Compute $\int_{-2}^{1} 3\,dx$ = Area of the Rectangle

$= (3)(3) = 9$



In general, if $k$ is a constant



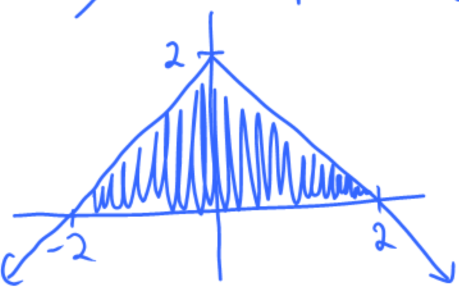

$$\int_a^b k\,dx = k(b-a)$$

Ex2/ Compute $\int_{-2}^{2} (2-|x|)\,dx$ = Area of the triangle

$= \frac{1}{2}(\text{Base})(\text{Height})$

$= \frac{1}{2}(4)(2)$

$= 4$




Ex3/ Compute $\int_0^3 (x-1)\,dx$ = Area of Blue Triangle − Area of Red Triangle

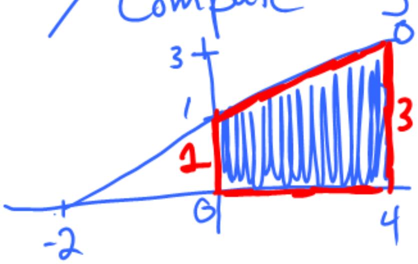
$= \frac{1}{2}(2)(2) - \frac{1}{2}(1)(1)$

$= 2 - \frac{1}{2}$

$= \frac{3}{2}$


Ex4/ Compute $\int_0^4 \left(\frac{1}{2}x+1\right)dx$


$A = \frac{1}{2}(b_1+b_2)h$

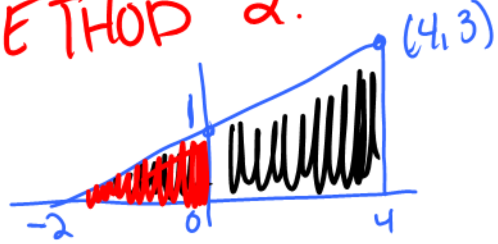



METHOD 1:

$\int_0^4 \left(\frac{1}{2}x+1\right)dx$ = Area of the trapezoid

$= \frac{1}{2}(1+3)(4) = 8$

METHOD 2:




$\int \left(\frac{1}{2}x+1\right)dx$ = Area of Black Triangle − Area of Red Triangle

$A = \frac{1}{2}(6)(3) - \frac{1}{2}(2)(1) = \frac{18}{2} - 1 = 9 - 1 = 8$

**Remark**

$$\int_0^4 \left(\tfrac{1}{2}x+1\right)dx = \int_{-2}^4 \left(\tfrac{1}{2}x+1\right)dx - \int_{-2}^0 \left(\tfrac{1}{2}x+1\right)dx$$

$$\int_{-2}^4 \left(\tfrac{1}{2}x+1\right)dx = \int_{-2}^6 \left(\tfrac{1}{2}x+1\right)dx + \int_0^4 \left(\tfrac{1}{2}x+1\right)dx$$

Ex 5/ Compute $\int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4}[\text{Area of full circle}]$

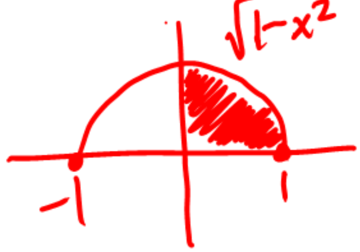
$$= \frac{1}{4}\left[\pi r^2\right]$$

$$= \frac{1}{4}\left[\pi \cdot 1^2\right] = \frac{\pi}{4}$$

$y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1$




Ex 6/ Compute $\int_{-2}^2 \sqrt{1 - \frac{x^2}{2^2}}\,dx$

$y = \sqrt{1 - \frac{x^2}{2^2}}$

$y^2 = 1 - \frac{x^2}{2^2}$

$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$

$a=2$ $b=1$

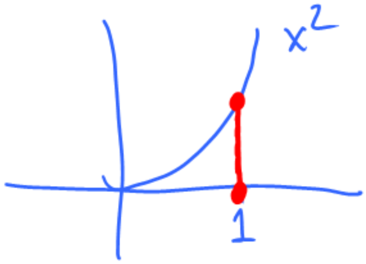

You will see in MA16020 that the area of an ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\pi ab$

So, $\int_{-2}^2 \sqrt{1 - \frac{x^2}{2^2}}\,dx = \frac{1}{2}[\text{Area of Ellipse}] = \frac{1}{2}\pi(2)(1)$

$$= \pi$$

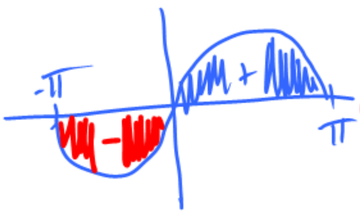
Ex 7/ Compute $\int_1^1 x^2\,dx$ = Area of a Line Segment

$= 0$




In general, $\int_a^a f(x)\,dx = 0$

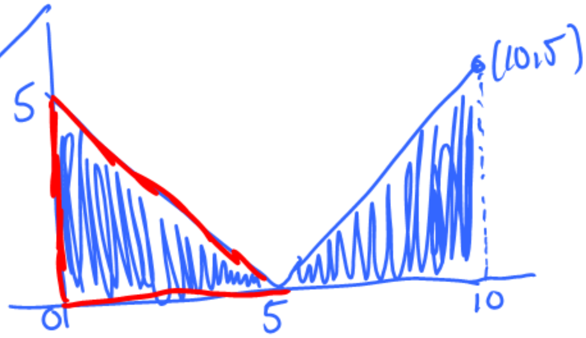
Ex 8/ Compute $\int_{-\pi}^{\pi} \sin(x)\,dx = 0$




← Matching Halves but opposite signs

Ex 9/

(a) $\int_0^{10} |x-5|\,dx$

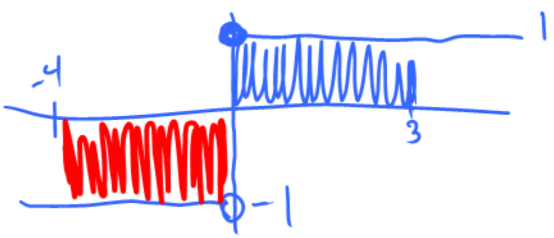



(b) $\int_0^{10} |x-5|\,dx$

$= \frac{1}{2}(5)(5) + \frac{1}{2}(5)(5)$

$= 25$

Ex 10/ $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

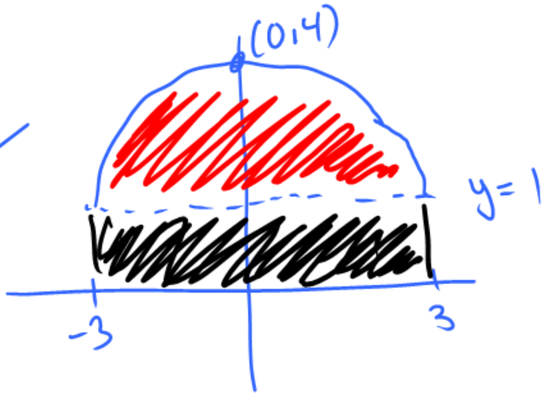



(a) $\int_{-4}^{3} f(x)\,dx$

(b) Area of Blue Rect. − Area of Red Rectangle

$$\int_{-1}^{3} f(x)\,dx = 3(1) - 4(1) = -1$$

Ex 11




(a) $$\int_{-3}^{3} \left( 1 + \sqrt{3^2 - x^2} \right) dx$$

(b)

$$\int_{-3}^{3} \left( 1 + \sqrt{9 - x^2} \right) dx = 6(1) + \frac{1}{2}\pi 3^2 = 6 + \frac{9}{2}\pi$$

**Remark** $$\int_{-3}^{3} \left( 1 + \sqrt{9 - x^2} \right) dx = \int_{-3}^{3} 1\,dx + \int_{-3}^{3} \sqrt{9 - x^2}\,dx$$