GOAL: Interpret the definite integral as the (signed) area underneath the graph of a function.
Link for Desmos Presentation: here

Recall Left/Right Riemann Sums (L_N and R_N) are used to approximate the area underneath the graph of a fcn.

Q: What happens when we take $N \to \infty$?

$$\lim_{N\to\infty} L_N = \lim_{N\to\infty} R_N = \text{The exact area}$$

Def The (definite) integral of a fcn f from a to b is

$$\int_a^b f(x)\,dx \overset{\text{def}}{=} \text{The signed area between the graph of } f \text{ and the } x\text{-axis on } [a,b].$$

$b \leftarrow$ Upper Limit

a: Lower Limit

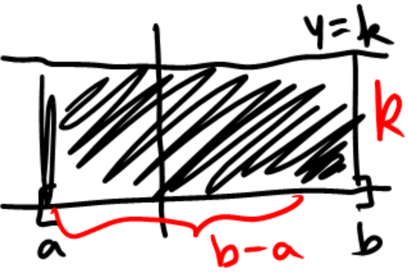
$$= \lim_{N\to\infty} L_N = \lim_{N\to\infty} R_N$$

a and b are called the limits (or bounds) of integration. If $\lim_{N\to\infty} L_N$ exists, we say f is Riemann integrable on $[a,b]$.

Ex/ Compute $\int_{-1}^{2} 3\,dx$ = Area of Rectangle

= (Base) (Height) = $3^2 = 9$

In general, if k is a constant

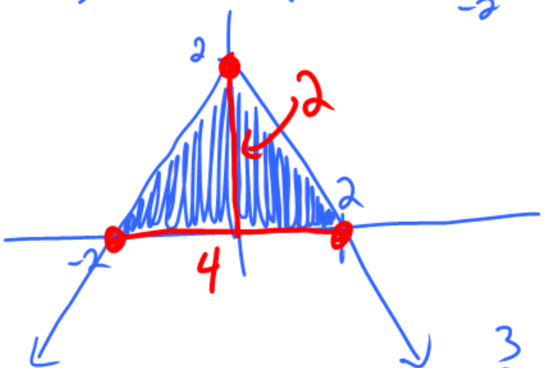


$$\int_a^b k \, dx = k(b-a)$$

Ex2/ Compute $\displaystyle\int_{-2}^{2} (2-|x|)\,dx$ = Area of Triangle

$= \frac{1}{2}(\text{Base})(\text{Height})$

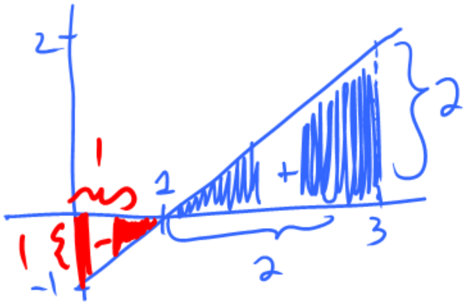
$= \frac{1}{2}(4)(2) = 4$



Ex3/ Compute $\displaystyle\int_0^3 (x-1)\,dx$ = Area of Blue Triangle − Area of Red Triangle

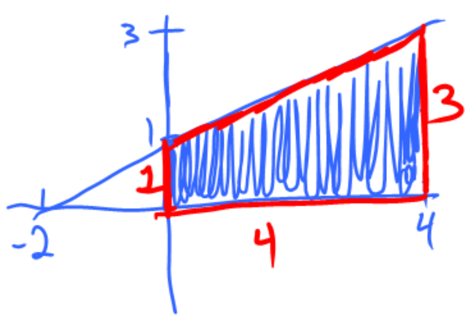
$= \frac{1}{2}(2)(2) - \frac{1}{2}(1)(1)$

$= 2 - \frac{1}{2} = \boxed{\frac{3}{2}}$



Ex4/ Compute $\displaystyle\int_0^4 \left(\tfrac{1}{2}x+1\right) dx$

$A = \frac{1}{2}(b_1+b_2)h$

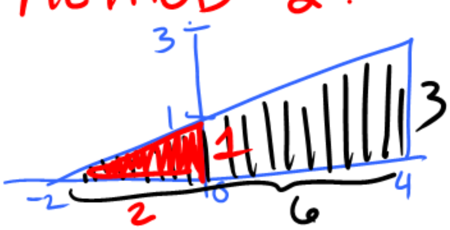


METHOD 1:

$\displaystyle\int_0^4 \left(\tfrac{1}{2}x+1\right) dx$ = Area of Trapezoid

$= \frac{1}{2}(1+3)(4) = 8$

METHOD 2:



$\displaystyle\int_0^4 \left(\tfrac{1}{2}x+1\right) dx$ = Area of Black Triangle − Area of Red Triangle

$$= \frac{1}{2}(6)(3) - \frac{1}{2}(2)(1) = 9 - 1 = 8$$

Remark

$$\int_0^4 \left(\tfrac{1}{2}x+1\right) dx = \int_{-2}^4 \left(\tfrac{1}{2}x+1\right) dx - \int_{-2}^0 \left(\tfrac{1}{2}x+1\right) dx$$

$$\int_{-2}^4 \left(\tfrac{1}{2}x+1\right) dx = \int_{-2}^0 \left(\tfrac{1}{2}x+1\right) dx + \int_0^4 \left(\tfrac{1}{2}x+1\right) dx$$

Ex 5/ Compute $\int_0^1 \sqrt{1-x^2}\, dx = \frac{1}{4}\left[\text{Area of full circle}\right]$

$$= \frac{1}{4}(\pi r^2)$$

$$= \frac{1}{4}(\pi \cdot 1^2) = \frac{\pi}{4}$$

$y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1$

Ex 6/ Compute $\int_{-2}^{2} \sqrt{1 - \frac{x^2}{2^2}}\, dx$

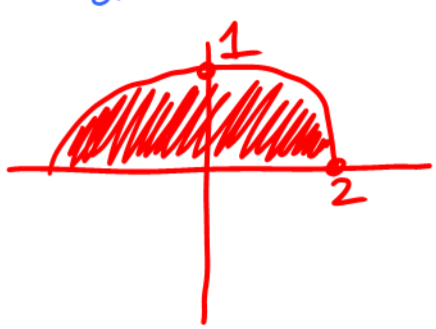
You will see in MA 16020 that the area of an ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab

$y = \sqrt{1 - \frac{x^2}{2^2}}$

$y^2 = 1 - \frac{x^2}{2^2}$

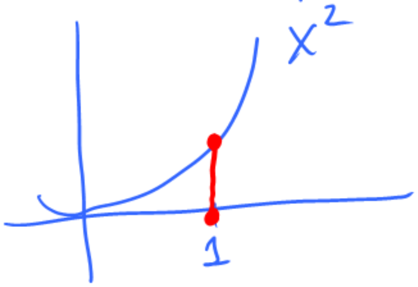
$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$

$a = 2 \quad b = 1$



So, $\int_{-2}^{2} \sqrt{1 - \frac{x^2}{2^2}}\, dx = \frac{1}{2}\left[\text{Area of Ellipse}\right] = \frac{1}{2}\pi(2)(1) = \pi$

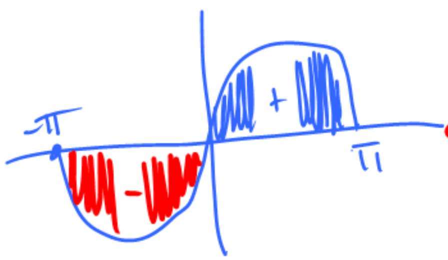
Ex 7/ Compute $\int_1^1 x^2\,dx$



= Area of the Line Segment = 0

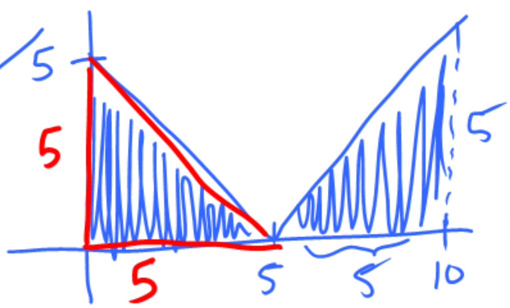
In general, $\int_a^a f(x)\,dx = 0$

Ex 8/ Compute $\int_{-\pi}^{\pi} \sin(x)\,dx = 0$



← Matching Halves, opposite signs

Ex 9/



ⓐ $\int_0^{10} |x-5|\,dx$

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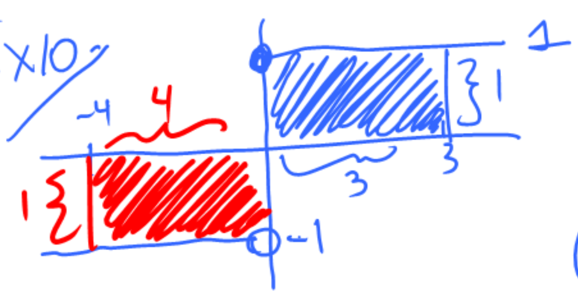
$= \frac{1}{2}(5)(5) + \frac{1}{2}(5)(5) = 25$

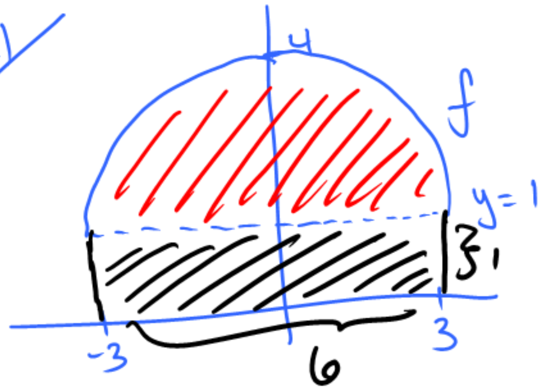
Ex10/



ⓐ $\int_{-4}^{3} f(x)\,dx$ where $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

ⓑ $\int_{-4}^{3} f(x)\,dx = 3(1) - 4(1) = -1$

~~Ex 11~~



ⓐ $\displaystyle\int_{-3}^{3}\left(1+\sqrt{3^2-x^2}\right)dx$

ⓑ = Area of Rect. + Area of Semi circle

$$= 6(1) + \frac{1}{2}\pi[3]^2 = 6 + \frac{9}{2}\pi$$

Remark $\displaystyle\int_{-3}^{3}\left(1+\sqrt{9-x^2}\right)dx = \int_{-3}^{3} 1\,dx + \int_{-3}^{3}\sqrt{9-x^2}\,dx$