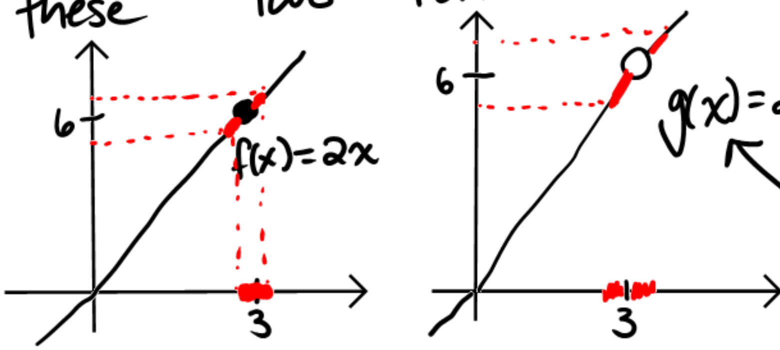
Lecture 2: Intro to Limits

Compare these two functions at $x=3$



$g(x)=2x\frac{(x-3)}{(x-3)}$

$\lim_{x\to 3} g(x)=6$

★ You will need a calculator today

We say <u>x approaches a</u> (written $x\to a$) to mean "the value of x gets really really close to a."

As $x\to 3$, $f(x)\to 6$ AND $g(x)\to 6$ even though $g(3)$ is undefined.

<u>Def</u> We write $\boxed{\lim_{x\to a} f(x)=L}$ as shorthand for "as $x\to a$, $f(x)\to L$."

L is called the <u>limit</u> of $f(x)$ as $x\to a$.

Ex/ Find a plausible value for

$$\lim_{x\to 4}\left(2\sqrt{x}-1\right)$$

← NOTE: Parantheses are needed for expressions with 2+ terms

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$2\sqrt{x}-1$	2.9497	2.995	2.9995		3.0005	3.005	3.0497

$\lim_{x\to 4}(2\sqrt{x}-1)=3$

Ex/ Find a plausible value for $\lim_{x\to 0}\frac{\sin x}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$\frac{\sin x}{x}$	0.9983	0.99998	0.99999		0.99999	0.99998	0.9983

$\lim_{x\to 0}\frac{\sin x}{x}=1$

Here are some limits that will need memorized

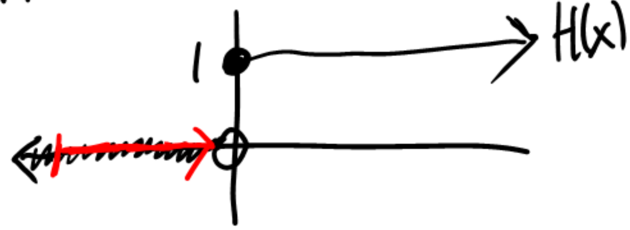
$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \; ; \; \lim_{x\to 0} \frac{\cos x - 1}{x} = 0 \; ; \; \lim_{x\to 0} \frac{e^{x}-1}{x} = 1$$

One-Sided Limits

Q: Can we always find a limit?

A: No, consider the Heaviside function

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



Around $x=0$,

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$H(x)$	0	0	0		1	1	1

$H(x)$ will not approach a particular value, so we say $\lim_{x\to 0} H(x)$ <u>does not exist</u> (abbreviated DNE)

<u>Def</u> The <u>left-sided limit</u> is the value (L) where $f(x) \to L$ as $x \to a$ from the left. This is written

$$\lim_{x\to a^-} f(x) \qquad \text{Ex/ } \lim_{x\to 0^-} H(x) = 0$$

Similiarly, the <u>right-sided limit</u>, written $\lim_{x\to a^+} f(x) = L$ means "$f(x) \to L$ as $x \to a$ from the right"

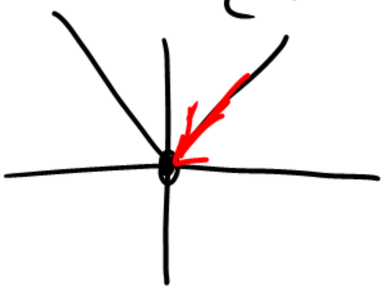
Ex/ $\lim_{x\to 0^+} H(x) = 1$

Remark ① If $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$, then $\lim_{x\to a} f(x) = L$

② If $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$, then $\lim_{x\to a} f(x)$ DNE

Ex of ①

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



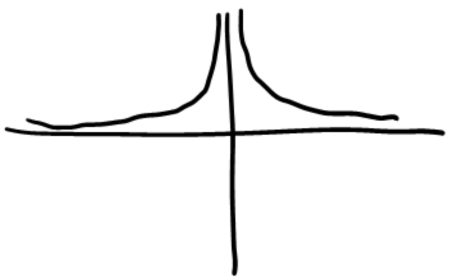
$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

So, $\lim_{x\to 0} |x| = 0$

Infinite Limits

Ex/ $f(x) = \frac{1}{x^2}$



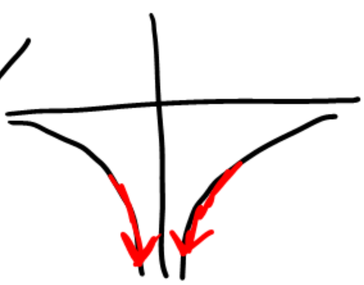
x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$\frac{1}{x^2}$	100	10000	10^6		10^6	10000	100

$$\frac{1}{\left(\frac{1}{10}\right)^2} = \frac{1}{\frac{1}{100}} = 100 \qquad \frac{1}{\left(\frac{1}{1000}\right)^2} = 10^6$$

$$\frac{1}{\left(\frac{1}{100}\right)^2} = \frac{1}{\frac{1}{10000}} = 10000$$

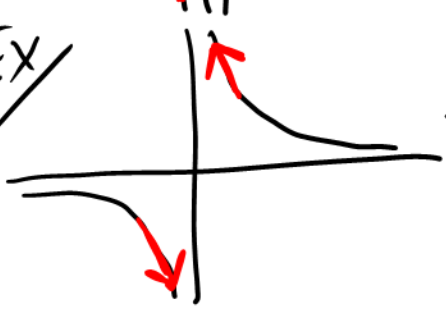
$\lim_{x\to 0} \frac{1}{x^2} = \infty$, ie, as $x \to 0$, $\frac{1}{x^2}$ grows without bound (goes to ∞)

Ex/



$f(x) = \frac{-1}{x^2}$, then $\lim_{x\to 0} \frac{-1}{x^2} = -\infty$

Ex/



$f(x) = \frac{1}{x}$

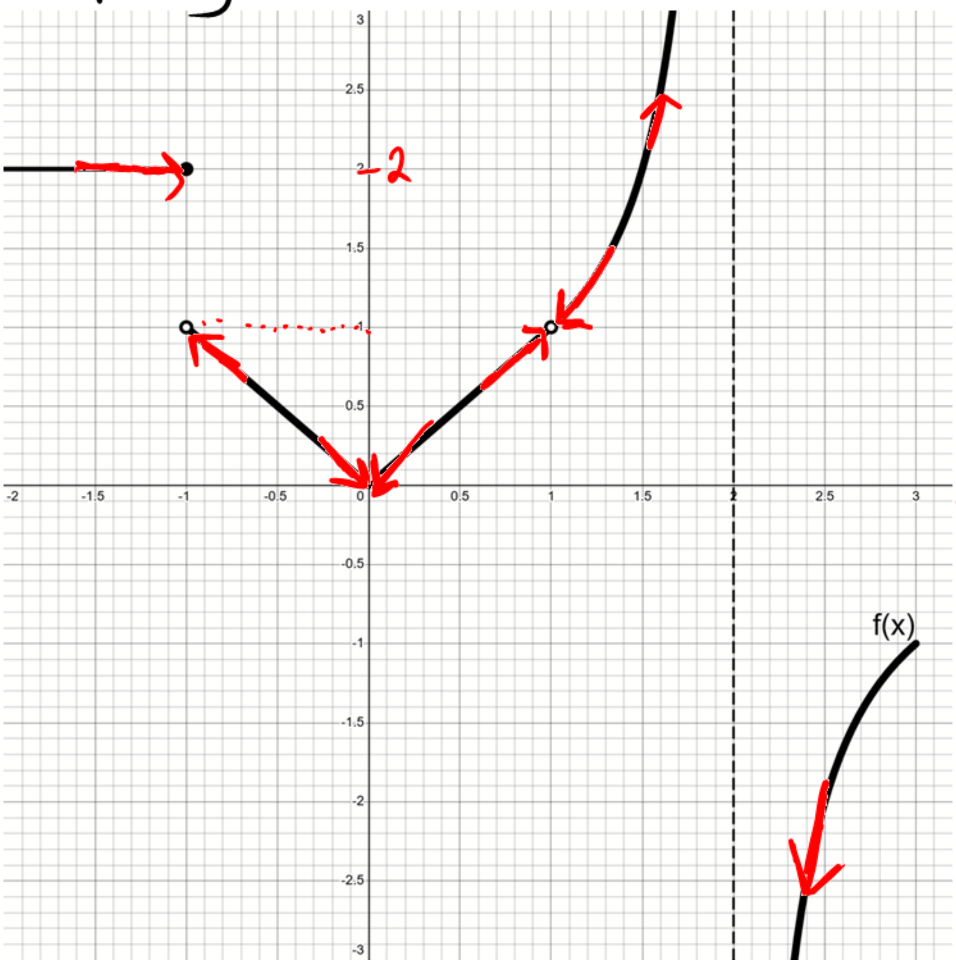
$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

$\lim_{x\to 0} \frac{1}{x}$ DNE

NOTE: If $\lim_{x \to a} f(x) = \infty$ or $-\infty$, technically the limit $\lim_{x \to a} f(x)$ DNE. But, we are more specific to why the limit DNE.

Computing Limits Graphically



$\lim_{x \to -1^-} f(x) = 2$

$\lim_{x \to -1^+} f(x) = 1$

$\lim_{x \to -1} f(x)$ DNE

$\lim_{x \to 0} f(x) = 0$

$\lim_{x \to 1} f(x) = 1$

$\lim_{x \to 2^-} f(x) = \infty$

$\lim_{x \to 2^+} f(x) = -\infty$