GOAL: Note how to manipulate expressions with definite integrals.

Recall The definite integral $\int_a^b f(x)\,dx$ measures the signed area underneath the graph of f on $[a,b]$.

Properties from Last Lecture:

1. $\int_a^b k\,dx = k(b-a)$, where k is a constant
2. $\int_a^a f(x)\,dx = 0$
3. If f is an $\underbrace{\text{odd function}}_{\text{e.g., } \sin(x)}$, then $\int_{-a}^{a} f(x)\,dx = 0$

Theorem Definite Integrals are linear:

1. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
2. $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$

Ex/ Suppose $\int_0^1 x^2\,dx = \frac{1}{3}$. Then

1. $\int_0^1 6x^2\,dx = 6\int_0^1 x^2\,dx = 6\left(\frac{1}{3}\right) = 2$
2. $\int_0^1 (4+3x^2)\,dx = \int_0^1 4\cdot 1\,dx + \int_0^1 3x^2\,dx$

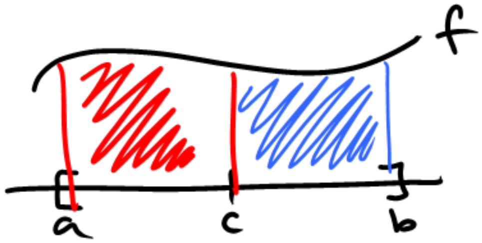
$$= 4\int_0^1 1\,dx + 3\int_0^1 x^2\,dx$$

$$= 4(1-0) + 3\left(\tfrac{1}{3}\right) = 4+1 = \boxed{5}$$

Theorem (Additivity) Let c be any real number

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Why?



Ex2/ Suppose $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$

$$\int_0^{10} f(x)\,dx = \int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx$$

$$17 = 12 + \int_8^{10} f(x)\,dx$$

$$\int_8^{10} f(x)\,dx = 5$$

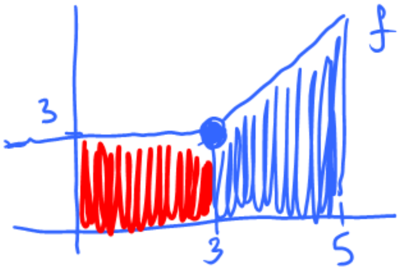
Ex3/ Suppose $\int_1^5 f(x)\,dx = 12$ and $\int_4^5 f(x)\,dx = 3.6$, compute $\int_1^4 f(x)\,dx$

$$\int_1^5 f(x)\,dx = \int_1^4 f(x)\,dx + \int_4^5 f(x)\,dx$$

$$12 = \int_1^4 f(x)\,dx + 3.6$$

$$\int_1^4 f(x)\,dx = 12 - 3.6 = 8.4$$

Ex 4/ Compute $\int_0^5 f(x)\,dx$ if $f(x) = \begin{cases} 3 & x < 3 \\ x & x \geq 3 \end{cases}$



$$\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx$$

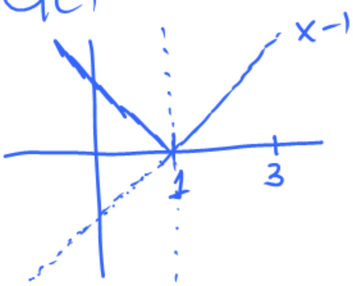
$$= \int_0^3 3\,dx + \int_3^5 x\,dx$$

$$= 3\cdot 3 + \frac{1}{2}(3+5)(2)$$

$$= 9 + 8 = \boxed{17}$$

Ex 5 Recall $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \Rightarrow |f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$

Get rid of the abs. value in



$$\int_0^3 |x-1|\,dx$$

$$|x-1| = \begin{cases} x-1 & x-1 \geq 0 \\ -(x-1) & x-1 < 0 \end{cases} = \begin{cases} x-1 & x \geq 1 \\ -(x-1) & x < 1 \end{cases}$$

$$\int_0^3 |x-1|\,dx = \int_0^1 |x-1|\,dx + \int_1^3 |x-1|\,dx$$

$$= \underbrace{\int_0^1 -(x-1)\,dx} + \underbrace{\int_1^3 (x-1)\,dx}$$

$$= \frac{1}{2} + 2 = \frac{5}{2}$$

Ex 6/ For $\int_a^b f(x)\,dx$, what's $\int_b^a f(x)\,dx$?

$$\int_a^b f(x)\,dx + \underbrace{\int_b^a f(x)\,dx}_{\int_a^b f(x)dx = -\int_b^a f(x)dx} = \int_a^a f(x)\,dx = 0$$

Theorem $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Ex 7/ If $\int_1^4 f(x) = 3$, then $\int_4^1 f(x)\,dx = -3$

Ex 8/ Rewrite the following as a single integral

$$\underbrace{\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx}_{} - \int_{-2}^{-1} f(x)\,dx$$

$$\int_{-2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

$$\int_{-2}^{5} f(x)\,dx - \left(-\int_{-1}^{-2} f(x)\,dx\right)$$

$$\int_{-2}^{5} f(x)\,dx + \int_{-1}^{-2} f(x)\,dx$$

$$\int_{-1}^{5} f(x)\,dx$$

Ex 9 (HW 30, Q5) Given $a < b < c$ and

$$\int_a^b g(x)\,dx = 3 \quad \text{and} \quad \int_a^c g(x)\,dx = 9\underbrace{\int_a^b g(x)\,dx}_{3}$$

Find $\int_b^c g(x)\,dx$

$$\int_a^b g(x)\,dx = \int_a^c g(x)\,dx + \int_c^b g(x)\,dx$$

$$3 = 27 + \int_c^b g(x)\,dx$$

$$\int_c^b f(x)\,dx = -24$$

$$-\int_b^c f(x)\,dx = -24 \implies \int_b^c f(x)\,dx = 24$$

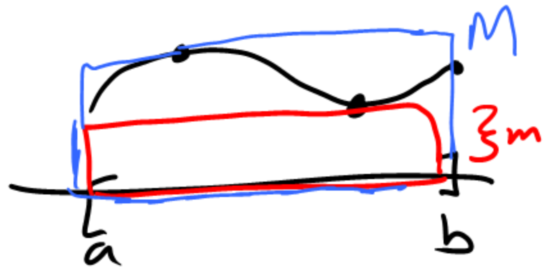
Non- Examinable

Theorem (Comparisons) Let f, g be continuous $[a,b]$

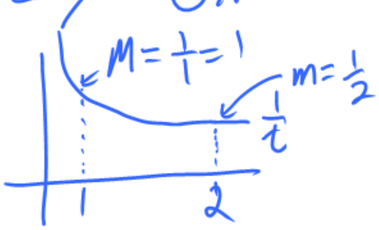
1. $f \geq 0$ on $[a,b]$, $\int_a^b f(x) \geq 0$
2. $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
3. $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$
4. (ML- Inequality) If M and m are the max and min of f on $[a,b]$

$$\underline{m(b-a)} \leq \int_a^b f(x)\,dx \leq \underline{M(b-a)}$$

Why (4)?



Ex 10) Give a crude estimate of $\ln(2) = \int_1^2 \frac{1}{t}\,dt$



$$\frac{1}{2} \le \ln(2) \le 1$$

$$\ln(2) \approx \frac{1+\frac{1}{2}}{2} = 0.75$$

$\ln(2) \approx 0.693 \leftarrow$ More Precise Answer via Calculator