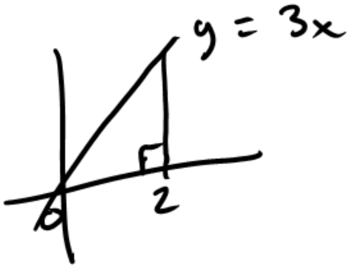
GOAL: Compute integrals of more complicated functions.

<u>Recall</u> Our main tool for computing integrals so far has been geometry.



$$\int_0^2 3x\,dx = \tfrac{1}{2}(2)(6) = 6$$

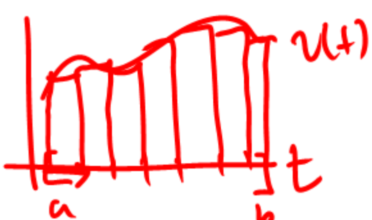
We want to way to do this with more complicated fcns. To do so, we use the <u>Fundamental Theorem of Calculus</u>

<u>Theorem (FTC)</u> Let f be continuous on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f.

<u>Why?</u> Think of it in terms of velocity and position



$$\int_{\text{START}}^{\text{END}} v(t)\,dt = \underset{\text{[Displacement]}}{\text{Change in Position}} = s(\text{END}) - s(\text{START})$$

Ex1/ Verify $\int_0^2 3x\,dx = 6$ using the FTC

$$\frac{x^{1+1}}{1+1} = \frac{1}{2}x^2$$

$$\int_0^2 3x\,dx = 3\int_0^2 x\,dx = 3\left[\tfrac{1}{2}(2)^2 - \tfrac{1}{2}(0)^2\right]$$

$$= 3[2-0] = 3\cdot 2 = 6$$

Ex2/ $\int_1^3 e^x\,dx = e^3 - e^1 = e^3 - e$

Remark 1 What happens when we pick another antiderivative? $e^x + C$

$$\int_1^3 e^x\,dx = (e^3 + C) - (e^1 + C) = e^3 - e$$

constants cancel. We often just choose the simplest one.

Remark 2 (Notation) We often write

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(x)\Big]_a^b = \Big[F(x)\Big]_a^b$$

$$= F(b) - F(a)$$

Ex3/ Find the area under the parabola $y = x^2$ from 0 to 1.

$$\int_0^1 x^2\,dx = \frac{x^{2+1}}{2+1}\Big|_0^1 = \frac{1}{3}x^3\Big|_0^1$$

$$= \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \frac{1}{3} - 0 = \frac{1}{3}$$

Ex4/ Compute $\displaystyle\int_3^6 \frac{dx}{x} = \int_3^6 \frac{1}{x}\,dx$

$$\int_3^6 \frac{1}{x}\,dx = \ln|x|\Big]_3^6 = \ln|6| - \ln|3| = \ln 6 - \ln 3$$

$$= \ln\frac{6}{3} = \ln 2$$

Ex5/ $$\int_0^{\pi/4} \cos x\,dx = \sin x\Big|_0^{\frac{\pi}{4}} = \sin\frac{\pi}{4} - \sin 0 = \frac{\sqrt{2}}{2} - 0$$

$$= \sqrt{2}/2$$

$$\int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

Ex 6 $\int_5^{10} \frac{1}{x^2} dx$

$$\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{-2+1}}{-2+1} + C = \frac{x^{-1}}{-1} + C$$

$$-\frac{1}{x} + C \leftarrow \text{Set } C = 0$$

$$\int_5^{10} \frac{1}{x^2} dx = -\frac{1}{x}\Big|_5^{10} = -\frac{1}{10} - \left(-\frac{1}{5}\right) = \frac{1}{5} - \frac{1}{10} = \frac{1}{10}$$

$$\frac{2}{10} - \frac{1}{10}$$

More Complicated Examples

Ex 7 $\int_0^1 (3 + x\sqrt{x})\, dx = \int_0^1 (3 + x^{\frac{3}{2}}) dx$

METHOD 1: Linearity

$$\int_0^1 (3 + x^{\frac{3}{2}}) dx = \int_0^1 3\, dx + \int_0^1 x^{\frac{3}{2}} dx$$

$$= 3\int_0^1 1\, dx + \int_0^1 x^{\frac{3}{2}} dx$$

$$\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} = \frac{2}{5} x^{\frac{5}{2}}$$

$$= 3[x]_0^1 + \left[\frac{2}{5} x^{\frac{5}{2}}\right]_0^1$$

$$= 3[1 - 0] + \left[\frac{2}{5} - 0\right] = 3 + \frac{2}{5} = \frac{17}{5}$$

METHOD 2: $\int_0^1 (3 + x^{\frac{3}{2}}) dx = \left[3x + \frac{2}{5} x^{\frac{5}{2}}\right]_0^1$

$$= \left[3 + \frac{2}{5}\right] - [0 + 0] = 3 + \frac{2}{5}$$

$$= \boxed{\frac{17}{5}}$$

Ex 8/ Compute $\int_1^2 \left(\frac{4+t^2}{t^3}\right) dt$

$$\int_1^2 \left(\frac{4+t^2}{t^3}\right)dt = \int_1^2 \left(\frac{4}{t^3}+\frac{t^2}{t^3}\right)dt = \int_1^2\left(\frac{4}{t^3}+\frac{1}{t}\right)dt$$

$\leftarrow \ln|t|$

$$4\int \frac{1}{t^3}\,dt = 4\int t^{-3}\,dt = 4\,\frac{t^{-3+1}}{-3+1} = \frac{4\cdot t^{-2}}{-2} = -\frac{2}{t^2}$$

Set $C=0$

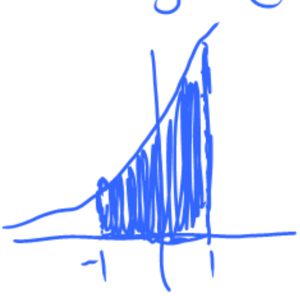
$$\int_1^2\left(\frac{4}{t^3}+\frac{1}{t}\right)dt = \left[\ln|t| - \frac{2}{t^2}\right]_1^2$$

$$= \left[\ln 2 - \frac{1}{2}\right] - \left[0-2\right]$$

$$= \ln 2 - \frac{1}{2} + 2 = \boxed{\ln 2 + \frac{3}{2}}$$

Often integration problems are phrased as follows, especially in MA16020 and MA261

Ex 9/ Find the area of the region bounded by $y=e^{x+1}$, $y=0$, $x=-1$, and $x=1$



$$\int_{-1}^{1} e^{x+1}\,dx = \int_{-1}^{1} e\cdot e^x\,dx = e\int_{-1}^{1} e^x\,dx$$

$$= e\left[e^x\right]_{-1}^{1} = e\left[e-\frac{1}{e}\right] = e^2-1$$

Ex 10/ $\int_0^{\pi/4} \sec x(\sec x + \cos x)\,dx = \int_0^{\pi/4}(\sec^2 x + 1)\,dx$

$$= \left[\tan x + x\right]_0^{\pi/4} = \left[\tan\frac{\pi}{4} + \frac{\pi}{4}\right] - \cancel{\left[\tan 0 + 0\right]}^{\,0}$$

$$= 1 + \frac{\pi}{4}$$

Ex 11 Find the area of the region bounded by $y = x^2 - x - 6$, $y = 0$, $x = -2$, and $x = 3$

$$\int_{-2}^{3}(x^2-x-6)\,dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x\right]_{-2}^{3}$$

$$= \left[9 - \frac{9}{2} - 18\right] - \left[-\frac{8}{3} - 2 + 12\right]$$

$$= \left[9 - 18 + 2 - 12\right] + \left[-\frac{9}{2} + \frac{8}{3}\right]$$

$$= -19 - \frac{11}{6} = -\frac{114}{6} - \frac{11}{6} = -\frac{125}{6}$$

$$\begin{array}{r} 19 \\ \times\ 6 \\ \hline 54 \\ 6\ \ \\ \hline 114 \end{array}$$

NOTE: $\int_a^b f(x)\,dx$ compute the <u>signed</u> area. To compute area, we need to compute $\int_a^b |f(x)|\,dx$

Ex 12 Consider the region bounded by $f(x) = 6x^3 - 6x^2 - 12x$, $y = 0$, $x = -1$, and $x = 2$

(a) Find net area

$$\int_{-1}^{2}(6x^3 - 6x^2 - 12x)\,dx = \left[\frac{3}{2}x^4 - 2x^3 - 6x^2\right]_{1}^{2}$$

$$= \left[24 - 16 - 24\right] - \left[\frac{3}{2} + 2 - 6\right] = -\frac{27}{2}$$

(b) Find area

$$\int_{-1}^{2}|f(x)|\,dx = \int_{-1}^{0}(6x^3 - 6x^2 - 12x)\,dx + \int_{0}^{2}(-6x^3 + 6x^2 + 12x)\,dx = \frac{5}{2} + 16$$

$$= 37/2$$