Lecture 32: The Net Change Theorem

GOAL: Apply the FTC to problems in the sciences.

Theorem (Net Change Theorem)

The total change along an interval is the net change

$$\underbrace{\int_a^b f'(t)\,dt}_{\text{"Adding up" contributions of a rate of change}} = \underbrace{f(b)-f(a)}_{\text{Net Change}}$$

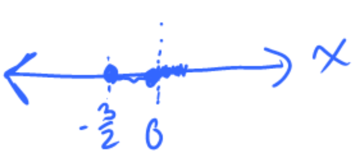
$$f(b) = f(a) + \int_a^b f'(t)\,dt$$

Displacement = Net Change in Position on $[a,b]$ = $\int_a^b v(t)\,dt$

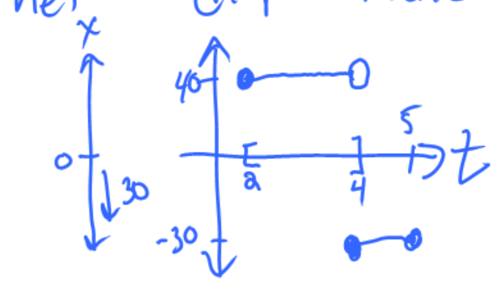
Ex) Let $v(t) = (3t-5)\frac{m}{s}$ be the velocity of an object. Find the displacement after 3 seconds.

$$s(3) - \underbrace{s(0)}_{=0} = \int_0^3 (3t-5)\,dt = \left[3\cdot\tfrac{1}{2}t^2 - 5t\right]_0^3$$

$$= \left[\frac{27}{2} - 15\right] - 0 = -\frac{3}{2}\text{ m}$$



Ex 2) A car moves North @ 40mph from 2-4pm, then moves South @ 30 mph from 4-5 pm. Find the net displacement



$$v(t) = \begin{cases} 40 & 2 \le t < 4 \\ -30 & 4 \le t \le 5 \end{cases}$$

Part b: Find Displacement

$$D = \int_2^5 v(t)\,dt = \int_2^4 v(t)\,dt + \int_4^5 v(t)\,dt$$

$$= \int_2^4 40\,dt + \int_4^5 (-30)\,dt$$

$$= \Big[40t\Big]_2^4 + \Big[-30t\Big]_4^5$$

$$= [160 - 80] + [-150 + 120]$$

$$= 80 - 30 = 50 \text{ miles North}$$

Ex3/ If a motor on a motor boat consumes gasoline at a rate $V'(t) = (5 - 0.1t^3)\,\frac{gal}{hr}$, how much gasoline was consumed in the first two hours?

Net Change in Volume $= V(2) - V(0) = \int_0^2 V'(t)\,dt$

$$\int_0^2 V'(t)\,dt = \int_0^2 (5 - 0.1t^3)\,dt = \Big[5t - 0.1 \cdot \tfrac{1}{4}t^4\Big]_0^2$$

$$= [10 - 0.4] - 0 = 9.6 \text{ gallons}$$

Ex4/ A population grows at a rate of $P'(t) = \sqrt{t}\,(100t + 5100)$ people per year. Find the net increase in the population from 4 years to 9 years

$$P(9) - P(4) = \int_4^9 P'(t)\,dt = \int_4^9 \sqrt{t}\,(100t + 5100)\,dt$$

$$= \int_4^9 \left(100t\sqrt{t} + 5100\sqrt{t}\right) dt = \int_4^9 \underbrace{\left(100t^{\frac{3}{2}} + 5100t^{\frac{1}{2}}\right)} dt$$

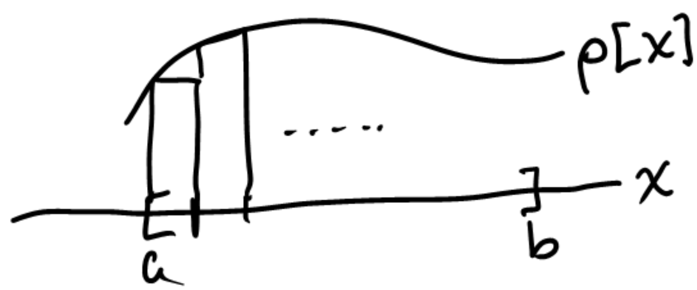
$$= \left[100 \cdot \frac{2}{5} t^{\frac{5}{2}} + 5100 \cdot \frac{2}{3} t^{\frac{3}{2}}\right]_4^9$$

$$= \left[40 \cdot 3^5 + 3400 \cdot 3^3\right] - \left[40 \cdot 2^5 + 3400 \cdot 2^3\right]$$

$$= 101,520 - 28,480 = 73,040 \text{ people}$$

Mass Mass = (Density) (Volume)

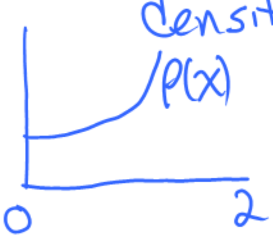
Suppose I have a thin rod with a density function $\rho(x)$ ↖ "rho"



The mass of a thin rod represented as an interval $[a,b]$ with density function $\rho(x)$ is

$$M = \int_a^b \rho(x)\,dx$$

Ex5/ Find the mass of a thin, 2m rod with density function $\rho(x) = (1+x^2)\ \frac{kg}{m}$



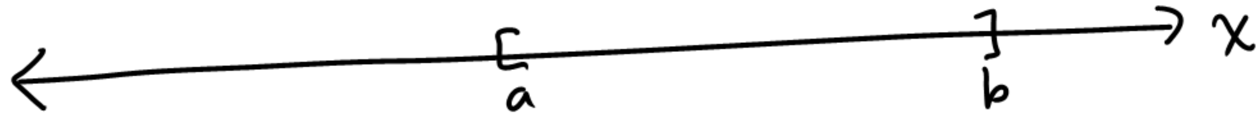
$$M = \int_0^2 \underbrace{(1+x^2)}_{} \underbrace{dx}_{m} = \left[x + \frac{1}{3}x^3\right]_0^2$$

$\frac{kg}{m} \cdot m$

$$= \left[2 + \frac{8}{3}\right] - [0 + 0]$$

$$= \frac{14}{3} \text{ kg}$$

Probability Loosely speaking, $P(a \leq x \leq b)$ is the likelihood that x is between a and b.



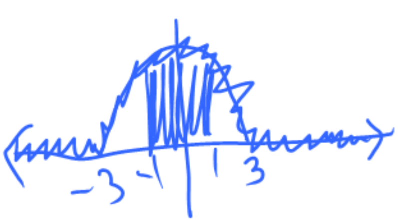
Def A continuous function f is called a Probability Density Function (PDF) if

① $f(x) \geq 0$

② $\int_{-\infty}^{\infty} f(x) = 1$

③ $P(a \leq x \leq b) = \int_a^b f(x)\, dx$

Ex 6 It can be shown that $f(x) = \begin{cases} \frac{1}{36}(9 - x^2) & \text{if } -3 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$ is a PDF. Compute $P(-1 \leq x \leq 1)$



$$P(-1 \leq x \leq 1) = \int_{-1}^{1} f(x)\, dx$$

$$= \int_{-1}^{1} \frac{1}{36}(9 - x^2)\, dx$$

$$\frac{1}{36} \int_{-1}^{1} (9 - x^2)\, dx = \frac{1}{36}\left[9x - \frac{1}{3}x^3\right]_{-1}^{1}$$

$$= \frac{1}{36}\left[\left(9-\frac{1}{3}\right)-\left(-9+\frac{1}{3}\right)\right] = \frac{1}{36}\left[18-\frac{2}{3}\right]$$

$$= \frac{1}{36}\left[\frac{52}{3}\right] = \frac{13}{27} \approx 48\%$$

Work

Work = (Force)(Distance)

Suppose I'm moving in a straight line with a force function $f(x)$

If an object travels along an interval $[a,b]$, the work done by the object is

$$W = \int_a^b f(x)\,dx$$

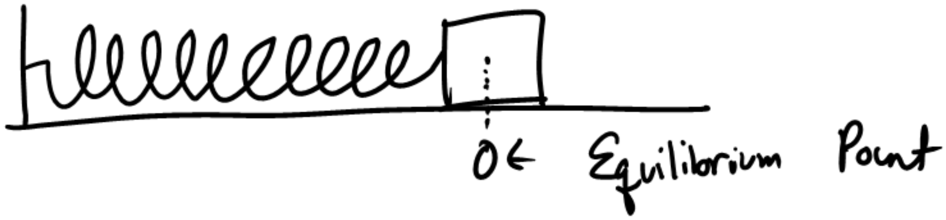
Ex 7/ When a particle is located a distance x from the origin, a force of (x^2+2x) N acts on it. How much work is done from $x=1$ to $x=3$?

$$W = \int_1^3 \underbrace{f(x)}_{N}\,\underbrace{dx}_{m} = \int_1^3 (x^2+2x)\,dx$$

$$= \left[\frac{1}{3}x^3 + x^2\right]_1^3$$

$$= [9+9] - \left[\frac{1}{3}+1\right] = 18 - \frac{4}{3} = \frac{50}{3} \text{ Nm}$$

$$= \frac{50}{3} \text{ J} \leftarrow \text{Joules}$$

Spring 

Hooke's Law The force required to keep a spring x units away from equilibrium is proportional to x

$$f(x) = kx$$

$k>0$ is called the spring constant.

Ex8/ It takes 40N to stretch a spring 0.05m past equilibrium. How much work is needed to stretch the spring from 0.05 m to 0.08 m past equilibrium

ⓐ Find Force function

$$f(x) = kx$$
$$40 = k(0.05)$$
$$k = \frac{40}{0.05} = 800 \frac{N}{m}$$

$$f(x) = 800x$$

ⓑ Find Work

$$W = \int_{0.05}^{0.08} f(x)\,dx = \int_{0.05}^{0.08} 800x\,dx = \left[400x^2\right]_{0.05}^{0.08}$$
$$= 400\left[0.08^2 - 0.05^2\right]$$
$$= 1.56 \text{ J}$$