GOAL: Apply the FTC to problems in the sciences.

Theorem (Net Change Theorem)

The total change along an interval is the net change

$$\underbrace{\int_a^b f'(t)\,dt}_{\text{"Sum" of a rate of change}} = \underbrace{f(b) - f(a)}_{\text{Net Change}} \qquad f(b) = f(a) + \int_a^b f'(t)\,dt$$

Displacement = Net Change in Position on $[a,b]$ $= \displaystyle\int_a^b v(t)\,dt$

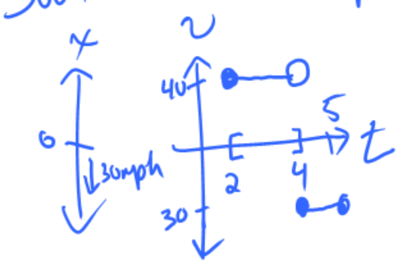
Ex1 Let $v(t) = (3t-5)\,\frac{m}{s}$ be the velocity of an object. Find its displacement after 3 seconds

$$\underbrace{S(3) - S(0)}_{\text{Assume } S(0)=0} = \int_0^3 \underbrace{v(t)}_{m/s}\underbrace{dt}_{s} = \int_0^3 (3t-5)\,dt$$

$$= \left[3\cdot\tfrac{1}{2}t^2 - 5t\right]_0^3 = \left[\tfrac{27}{2} - 15\right] - [0-0]$$

$$= -\frac{3}{2}\,m$$

Ex2 A car moves North @ 40mph from 2-4pm, the moves South @ 30mph from 4-5pm. Find the net displacement.



$$v(t) = \begin{cases} 40 & 2 \le t < 4 \\ -30 & 4 \le t \le 5 \end{cases} \qquad S(5) - S(2) = \int_2^5 v(t)\,dt$$

$$\int_2^5 \underbrace{v(t)}_{\frac{mi}{hr}}\, \underbrace{dt}_{hr} = \int_2^4 v(t)\, dt + \int_4^5 v(t)\, dt$$

$$= \int_2^4 40\, dt + \int_4^5 (-30)\, dt = \left[40t\right]_2^4 + \left[-30t\right]_4^5$$

$$= [160 - 80] + [-150 - (-120)] = 80 - 30$$

$$= 50$$ miles North

Ex3 If the motor on a motorboat consume gasoline at a rate of $V'(t) = (5 - 0.1t^3)\,\frac{gal}{hr}$, how much gas was consumed in the first 2 hours?

$$V(2) - V(0) = \int_0^2 \underbrace{V'(t)}_{\frac{gal}{hr}}\, \underbrace{dt}_{hr} = \int_0^2 (5 - 0.1t^3)\, dt$$

$$= \left[5t - 0.1 \cdot \tfrac{1}{4} t^4\right]_0^2$$

$$= [10 - 0.4] - [0 - 0] = 9.6$$ gallons

Ex4 A population grows at a rate of $P'(t) = \sqrt{t}\,(100t + 5100)$ people per year. Find the net increase in the population from 4 years to 9 years.

$$P(9) - P(4) = \int_4^9 P'(t)\, dt = \int_4^9 \sqrt{t}\,(100t + 5100)\, dt$$

$$= \int_4^9 (100t\sqrt{t} + 5700\sqrt{t})\, dt$$

$$= \int_4^9 \left(100t^{\frac{3}{2}} + 5100t^{\frac{1}{2}}\right) dt$$

$$= \left[100 \cdot \frac{2}{5} t^{\frac{5}{2}} + 5100 \cdot \frac{2}{3} t^{\frac{3}{2}}\right]_4^9 = \left[40t^{\frac{5}{2}} + 3400t^{\frac{3}{2}}\right]_4^9$$

$$= \left[40 \cdot 3^5 + 3400 \cdot 3^3\right] - \left[40 \cdot 2^5 + 3400 \cdot 2^3\right]$$

$$= 101{,}520 - 28{,}480 = 73{,}040 \text{ people}$$

Densities

Mass Density

Mass = (Density)(Volume)

Suppose I have a thin rod with density function $\rho(x)$:
"rho"

The density of a thin rod represented as an interval $[a,b]$ with density function $\rho(x)$ is

$$M = \int_a^b \rho(x)\, dx$$

Ex5/ Find the mass of a thin, 2m rod with density function

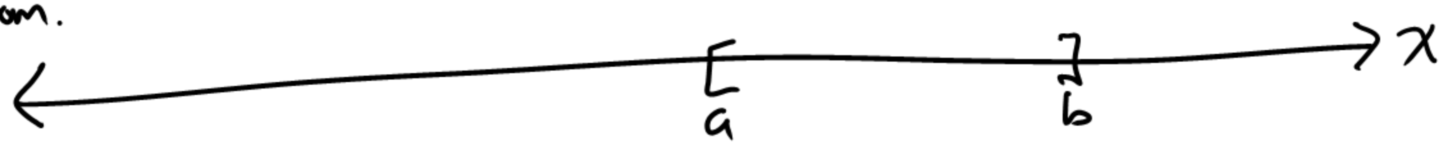
$\rho(x) = (1+x^2)\,\frac{kg}{m}$

$$M = \int_0^2 \underbrace{\rho(x)}_{\frac{kg}{m}}\, \underbrace{dx}_{m} = \int_0^2 (1+x^2)\, dx = \left[x + \frac{1}{3}x^3\right]_0^2$$

$$= \left[2 + \frac{8}{3}\right] - [0+0] = 2 + \frac{8}{3}$$

$$= \frac{14}{3} \text{ kg}$$

Probability Loosely speaking, $P(a \le x \le b)$ is the likelihood that x is between a and b if x is chosen at random.



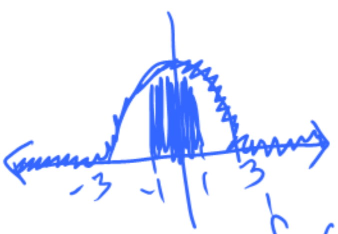
Def A continuous function f is called a **Probability Density Function** (PDF) if

① $f(x) \ge 0$

② $\int_{-\infty}^{\infty} f(x) = 1$

③ $P(a \le x \le b) = \int_a^b f(x)\,dx$

Ex 6 It can be shown that $f(x) = \begin{cases} \frac{1}{36}(9-x^2) & \text{if } -3 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$ is a PDF. Find $P(-1 \le x \le 1)$

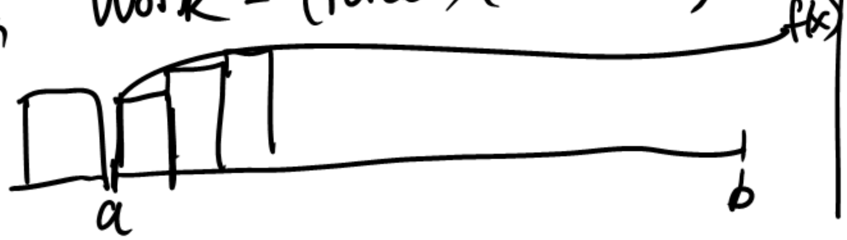


$$P(-1 \le x \le 1) = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \frac{1}{36}(9-x^2)\,dx$$

$$= \frac{1}{36}\int_{-1}^{1}(9-x^2)\,dx = \frac{1}{36}\left[9x - \frac{1}{3}x^3\right]_{-1}^{1}$$

$$= \frac{1}{36}\left[\left(9-\frac{1}{3}\right) - \left(-9+\frac{1}{3}\right)\right] = \frac{1}{36}\left[18 - \frac{2}{3}\right] = \frac{13}{27} \approx 48\%$$

Work Work = (Force)(Distance)



Suppose I have a force function $f(x)$

If an object travels along an interval $[a,b]$, the work done by the object is

$$W = \int_a^b f(x)\,dx$$

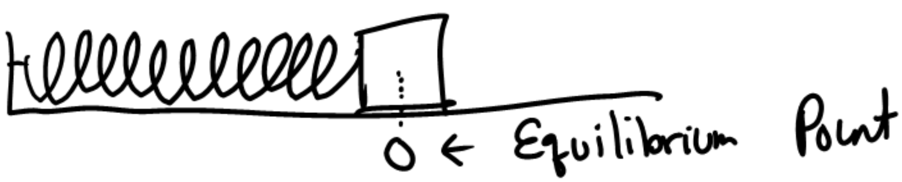
Ex 7/ When a particle is located x units away from the origin, a force of (x^2+2x) N acts on it. How much work is done from $x=1$ to $x=3$?

$$W = \int_1^3 \underbrace{(x^2+2x)}_{N}\,\underbrace{dx}_{m}$$

$$= \left[\tfrac{1}{3}x^3 + x^2\right]_1^3 = [9+9] - \left[\tfrac{1}{3}+1\right] = 18 - \tfrac{4}{3}$$

$$= \tfrac{50}{3}\,\text{Nm} = \tfrac{50}{3}\,J \leftarrow \text{Joules}$$

<u>Springs</u>



<u>Hooke's Law</u> The force required to keep a spring x units away from equilibrium is proportional to x.

$$f(x) = kx$$

$k>0$ is called the <u>spring constant</u>.

Ex 8/ It takes 40N to stretch a spring 0.05 m past equilibrium. How much work is needed to stretch the spring from 0.05 m to 0.08 m past equilibrium?

ⓐ Find Force function

$$f(x) = kx$$

$$40 = k(0.05)$$

$$k = \frac{40}{0.05} = 800 \frac{N}{m} \Rightarrow f(x) = 800x$$

ⓑ Find Work

$$W = \int_{0.05}^{0.08} 800x \, dx = \left[400x^2\right]_{0.05}^{0.08} = 400\left[0.08^2 - 0.05^2\right]$$

$$= 1.56 \text{ J}$$