GOAL: Approximate the value of an integral; typically used when the FTC is difficult (or impossible) to apply.

Q: How can we find the decimal approximation for $\ln 5$?

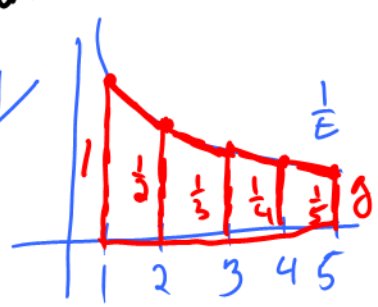
Recall $\int_1^5 \frac{1}{t}\,dt = \ln 5 - \ln 1 = \ln 5$

We have seen ways to approximate $\int_1^5 \frac{1}{t}\,dt$:

- $L_4 = \frac{25}{12} \approx 2.0833$
- $R_4 = \frac{77}{60} \approx 1.2833$
- (ML-Inequality): $\ln 5 \approx \frac{4+\frac{4}{5}}{2} = 2.4$ (Very Bad)

Can we do better?

Ex)



Make $N=4$, $\Delta x = \frac{5-1}{4} = 1$

$\int_1^5 \frac{1}{t}\,dt \approx \int_1^5 g(t)\,dt = \sum$ Area of Trapezoids

$A = \frac{1}{2}(b_1+b_2)h$

$$= \tfrac{1}{2}(1+\tfrac{1}{2})(1) + \tfrac{1}{2}(\tfrac{1}{2}+\tfrac{1}{3})(1) + \tfrac{1}{2}(\tfrac{1}{3}+\tfrac{1}{4})(1) + \tfrac{1}{2}(\tfrac{1}{4}+\tfrac{1}{5})(1)$$

$$= \frac{(1)}{2}\left[1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{5}\right]$$

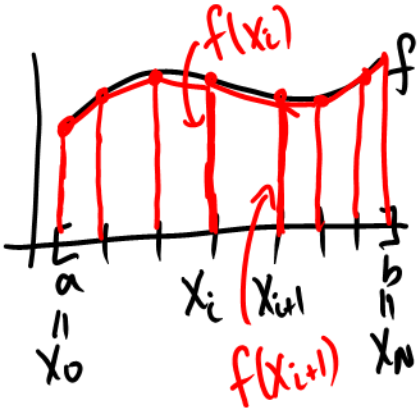
$$= \tfrac{1}{2}\left[1 + 2(\tfrac{1}{2}) + 2(\tfrac{1}{3}) + 2(\tfrac{1}{4}) + \tfrac{1}{5}\right]$$

$$= \tfrac{1}{2}\left[1 + 1 + \tfrac{2}{3} + \tfrac{1}{2} + \tfrac{1}{5}\right]$$

$$= \tfrac{1}{2}\left[\tfrac{101}{30}\right] = \tfrac{101}{60} \approx 1.68333$$

For context, $\ln 5 \approx 1.6094$

Q: How can we do this in general?



$$\Delta x = \frac{b-a}{\#\text{Trapezoids}} = \frac{b-a}{N}$$

$$x_i = a + i\Delta x$$

Area of each $\bigtriangleup$: $\frac{1}{2}[f(x_i) + f(x_{i+1})]\Delta x$

$$\int_a^b f(x)\,dx \approx \sum \text{Area of } \bigtriangleup\text{s} = \sum_{i=0}^{N} \frac{1}{2}[f(x_i) + f(x_{i+1})]\Delta x$$

$$= \frac{\Delta x}{2} \sum_{i=0}^{N} [f(x_i) + f(x_{i+1})]$$

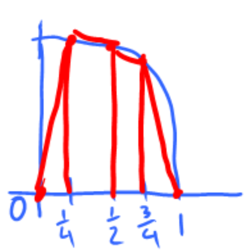
$$= \frac{\Delta x}{2}[f(a) + \underbrace{f(x_1) + f(x_1)} + f(x_2) + \cdots\cdots + f(x_{N-1}) + f(b)]$$

<u>Def</u> This process is called the Trapezoidal Rule with N rectangles

$$\int_a^b f(x)\,dx \approx T_N := \frac{\Delta x}{2}\left[f(a) + 2\sum_{i=1}^{N-1} f(x_i) + f(b)\right]$$

$\Delta x = \frac{b-a}{N}$ and $x_i = a + i\Delta x$

Ex 3 Note that $\pi = 4\int_0^1 \sqrt{1-x^2}\,dx$. Approximate the integral by using 4 trapezoids.



$$\Delta x = \frac{b-a}{N} = \frac{1-0}{4} = \frac{1}{4}$$

$$T_4 = \frac{1/4}{2}\left[f(0) + 2f(\tfrac{1}{4}) + 2f(\tfrac{1}{2}) + 2f(\tfrac{3}{4}) + f(1)\right]$$

$f(x) = \sqrt{1-x^2}$

$$T_4 = \frac{1}{8}\left[1 + 2\sqrt{1-\tfrac{1}{16}} + 2\sqrt{1-\tfrac{1}{4}} + 2\sqrt{1-\tfrac{9}{16}} + 0\right]$$

$$= \frac{1}{8}\left[1 + 1.93649 + 1.73205 + 1.32288\right]$$

$$= \frac{1}{8}\left[5.99142\right] = 0.7489275$$

So, $\pi \approx 4T_4 = 2.99571$. We can do better by adding more trapezoids

n	$\pi \approx 4T_n$
4	2.99571
100	3.14041

Q: Why would we do this?

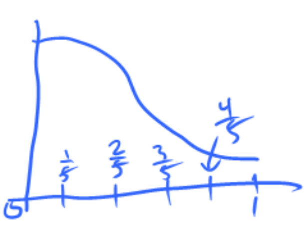
A1: Approximate values defined by integrals

$$\ln(x) \overset{def}{=} \int_1^x \frac{1}{t}\,dt \qquad \mathrm{erf}(x) \overset{def}{=} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

$$Li(x) \overset{def}{=} \int_2^x \frac{1}{\ln t}\,dt \qquad \pi = 12\left[-\frac{\sqrt{3}}{8} + \int_0^{1/2} \sqrt{1-x^2}\,dx\right]$$

A2: Evaluate integrals where FTC is hard/impossible to apply

Ex3/ Approximate $\int_0^1 e^{-x^2}\,dx$ via 5 trapezoids



$$\Delta x = \frac{1-0}{5} = \frac{1}{5}$$

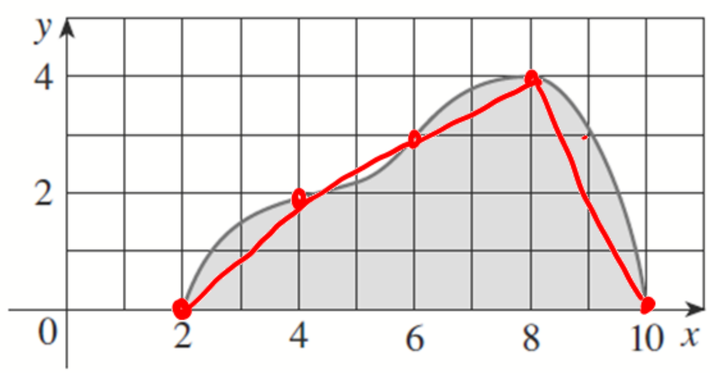
$$T_5 = \frac{1/5}{2}\left[f(0) + 2f\left(\tfrac{1}{5}\right) + 2f\left(\tfrac{2}{5}\right) + 2f\left(\tfrac{3}{5}\right) + 2f\left(\tfrac{4}{5}\right) + f(1)\right]$$

$$T_5 = \frac{1}{10}\left[1 + 2e^{-\frac{1}{25}} + 2e^{-\frac{4}{25}} + 2e^{-\frac{9}{25}} + 2e^{-\frac{16}{25}} + e^{-1}\right]$$

$$= \frac{1}{10}[7.44368] = 0.744368$$

A3: When you don't have a function to work with.

Ex4 A speedometer records the velocity of a car every second. Given the graph/data, approx. the displacement via 4 rectangles



$$\Delta x = \frac{10-2}{4} = 2$$

x	2	4	6	8	10
$f(x)$	0	2	3	4	0

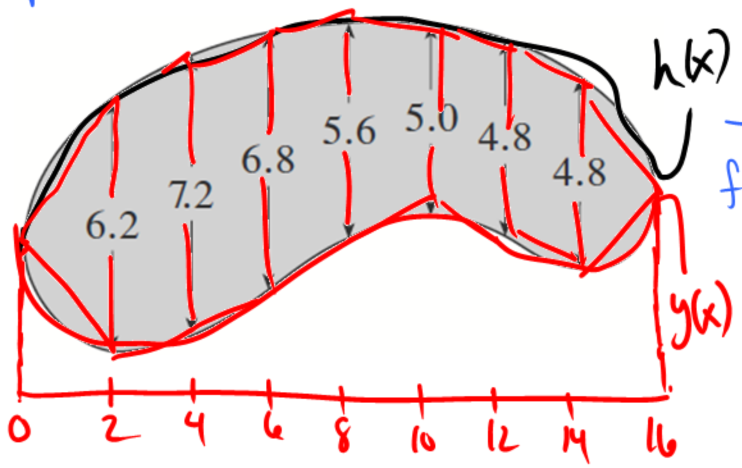
$$T_4 = \frac{2}{2}\left[f(2) + 2f(4) + 2f(6) + 2f(8) + f(10)\right]$$

$$= 0 + 2(2) + 2(3) + 2(4) + 0$$

$$= 4 + 6 + 8$$

$$= 18$$

Ex5/ Every 2 meters, the width of a kidney-bean shaped pool is measured. Approximate the area of the pool.



x	0	2	4	6	8	10	12	14	16
$f(x)$	0	6.2	7.2	6.8	5.6	5.0	4.8	4.8	0

Approximating $\int_0^{16}[h(x) - g(x)]\,dx$

$\Delta x = 2$

$$T_8 = \frac{2}{2}\left[f(0) + 2f(2) + 2f(4) + 2f(6) + 2f(8) + 2f(10) + 2f(12) + 2f(14) + f(16)\right]$$

$$= 0 + 2(6.2) + 2(7.2) + 2(6.8) + 2(5.6) + 2(5.0) + 2(4.8) + 2(4.8) + 0$$

$$= 80.8 m^2$$

$$= 80.8 m^2$$