GOAL: Approximate the value of an integral; typically used when the FTC is difficult (or impossible) to apply.

Q: How can we find the decimal approximation for $\ln 5$?

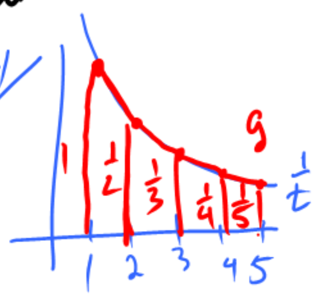
Recall $\int_1^5 \frac{1}{t}\,dt = \ln 5 - \ln 1 = \ln 5$

We have seen ways to approximate $\int_1^5 \frac{1}{t}\,dt$:

- $L_4 = \frac{25}{12} \approx 2.0833$
- $R_4 = \frac{77}{60} \approx 1.2833$
- (ML-Inequality): $\ln 5 \approx \frac{4 + \frac{4}{5}}{2} = 2.4$ (Very Bad)

Can we do better?

Ex/



$n = 4$ Subintervals, $\Delta x = \frac{5-1}{4} = 1$

$A = \frac{1}{2}(b_1 + b_2)h$

$\int_1^5 \frac{1}{t}\,dt \approx \int_1^5 g(t)\,dt = \sum$ Area of Trapezoids

$= \frac{1}{2}(1+\frac{1}{2})(1) + \frac{1}{2}(\frac{1}{2}+\frac{1}{3})(1) + \frac{1}{2}(\frac{1}{3}+\frac{1}{4})(1) + \frac{1}{2}(\frac{1}{4}+\frac{1}{5})(1)$

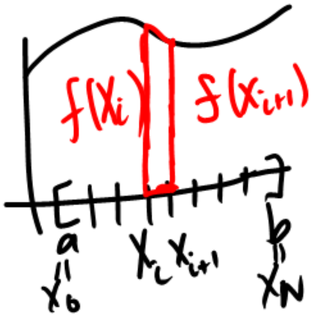
$= \frac{(1)}{2}\left[1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{5}\right]$

$= \frac{1}{2}\left[1 + 2(\frac{1}{2}) + 2(\frac{1}{3}) + 2(\frac{1}{4}) + \frac{1}{5}\right]$

$= \frac{1}{2}\left[1 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{5}\right] = \frac{1}{2}\left[\frac{101}{30}\right] = \frac{101}{60} \approx 1.6833$

For context, $\ln 5 \approx 1.6094$

Q: How can we do this in general.



$N = \#$ of trapezoids

$\Delta X = \dfrac{b-a}{N}$; $x_i = a + i\Delta X$

Area of a trapezoid : $\frac{1}{2}(f(X_i) + f(X_{i+1}))\Delta X$

$$\int_a^b f(x)\,dx \approx \sum \text{Area of Trapezoids} = \sum_{i=0}^{N-1} \frac{1}{2}\left[f(X_i) + f(X_{i+1})\right]\Delta X$$

$$= \frac{\Delta X}{2}\sum_{i=0}^{N-1}\left[f(X_i) + f(X_{i+1})\right] = \frac{\Delta X}{2}\left[f(a) + f(X_1) + f(X_1) + f(X_2) + f(X_2) + f(X_3) + \cdots + f(X_{N-1}) + f(X_{N-1}) + f(b)\right]$$

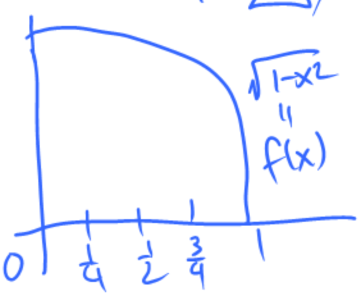
<u>Def</u> This process is called the Trapezoidal Rule with N trapezoids.

$$\int_a^b f(x)\,dx \approx T_N \stackrel{\text{def}}{=} \frac{\Delta X}{2}\left[f(a) + \sum_{i=1}^{N-1} f(X_i) + f(b)\right]$$

where $\Delta X = \dfrac{b-a}{N}$ and $X_i = a + i\Delta X$

EX2/ Note $\pi = 4\int_0^1 \sqrt{1-x^2}\,dx$. Approximate π via 4 trapezoids.

Find T_4: $\Delta X = \dfrac{1-0}{4} = \dfrac{1}{4}$



$$T_4 = \frac{1/4}{2}\left[f(0) + 2f(\tfrac{1}{4}) + 2f(\tfrac{1}{2}) + 2f(\tfrac{3}{4}) + f(1)\right]$$

$$T_4 = \frac{1}{8}\left[1 + 2\sqrt{1-\tfrac{1}{16}} + 2\sqrt{1-\tfrac{1}{4}} + 2\sqrt{1-\tfrac{9}{16}} + 0\right]$$

$$= \frac{1}{8}\left[1 + 1.93649 + 1.73205 + 1.32288 + 0\right]$$

$$= \frac{1}{8}[5.99142] = 0.7489$$

$\pi \approx 4T_4 = 2.99571$. Add more trapezoids to get a better approximation

N	$\pi \approx 4T_n$
4	2.99571
100	3.14041

Q: Why do we do this?

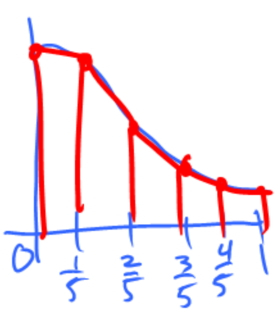
A1: Approximate values defined by integrals

$$\ln(x) \overset{\text{def}}{=} \int_1^x \frac{1}{t}\,dt \qquad \operatorname{erf}(x) \overset{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

$$Li(x) \overset{\text{def}}{=} \int_2^x \frac{1}{\ln t}\,dt \qquad \pi = 12\left[-\frac{\sqrt{3}}{8} + \int_0^{1/2} \sqrt{1-x^2}\,dx\right]$$

A2: Evaluate integrals where FTC is hard/impossible to apply

Ex3/ Approximate $\int_0^1 e^{-x^2}\,dx$ via 5 trapezoids



$$\Delta x = \frac{1-0}{5} = \frac{1}{5}$$

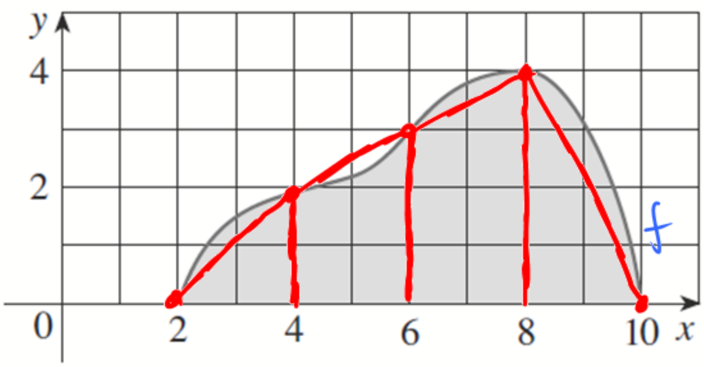
$$T_5 = \frac{1/5}{2}\left[f(0) + 2f\left(\tfrac{1}{5}\right) + 2f\left(\tfrac{2}{5}\right) + 2f\left(\tfrac{3}{5}\right) + 2f\left(\tfrac{4}{5}\right) + f(1)\right]$$

$$T_5 = \frac{1}{10}\left[1 + 2e^{-\frac{1}{25}} + 2e^{-\frac{4}{25}} + 2e^{-\frac{9}{25}} + 2e^{-\frac{16}{25}} + e^{-1}\right]$$

$$= \frac{1}{10}[7.44368] = 0.744368$$

A3: When we don't have a function to work with

Ex4/ A speedometer records the velocity of a car every second. Given the graph below, approximate disp. via 4 trapezoids



$\Delta x = \frac{10-2}{4} = 2$

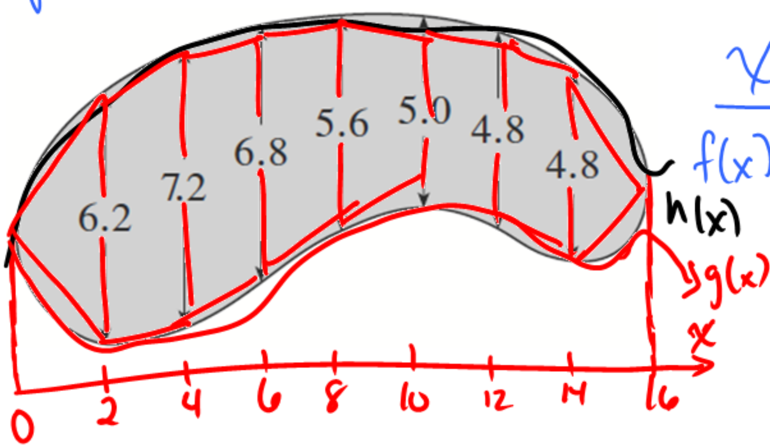
x	2	4	6	8	10
$f(x)$	0	2	3	4	0

$T_4 = \frac{2}{2}[f(2) + 2f(4) + 2f(6) + 2f(8) + f(10)]$

$= 0 + 2(2) + 2(3) + 2(4) + 0 = 4+6+8$

$= 18$

Ex5/ Every 2 meters, the width of a kidney-bean shaped pool is measured. Approximate the area of the pool.



x	0	2	4	6	8	10	12	14	16
$f(x)$	0	6.2	7.2	6.8	5.6	5	4.8	4.8	0

Approximate $\int_0^{16} [h(x) - g(x)]\, dx$

"

$\int_0^{16} f(x)\, dx$

$\Delta x = 2$, $N = 8$

$T_8 = \frac{2}{2}[f(0) + 2f(2) + 2f(4) + 2f(6) + 2f(8) + 2f(10) + 2f(12) + 2f(14) + f(16)]$

$$= 0 + 2(6.2) + 2(7.2) + 2(6.8) + 2(5.6) + 2(5) + 2(4.8) + 2(4.8) + 0$$

$$= 80.8\,m^2$$