Announcements

1. Exams are on the table
   - Get them; I do not want them

2. Final Exam Review all of next week:
   - Think about what topics/questions you want to see
   - I'll ask Friday

3. EC 4 Due this Sunday @ 11:59PM

4. Complete course evaluations on Brightspace

GOAL: Discuss situations governed by the equation $P(t) = P_0 e^{kt}$ for $k > 0$.

We want to study the differential equation

$$\frac{dy}{dt} = \underset{\text{constant}}{k}y$$

To solve, we use <u>separation of variables</u>.

$$\frac{dy}{dt} = ky$$

$$\frac{dy}{dt} = \frac{k}{\frac{1}{y}}$$

$$\frac{1}{y}\,dy = k\,dt$$

$$\int \frac{1}{y}\,dy = \int k\,dt$$

$$\ln|y| + C_1 = kt + C_2$$

$$\ln|y| = kt + [C_2 - C_1]$$

$$|y| = e^{kt + [C_2 - C_1]}$$

$$y = \underbrace{\boxed{\pm e^{C_2 - C_1}}}_{\text{Arbitrary Constant, call it } C} e^{kt}$$

Don't need to know this until MA16020

So the general solution is $y(t) = Ce^{kt}$

Q: What if we have an initial condition $y(0) = P_0$

$$y(t) = Ce^{kt}$$

$$P_0 \overset{\text{set}}{=} y(0) = Ce^{k \cdot 0}$$

$$C = P_0$$

<u>Theorem</u> The solution to the IVP

$$\begin{cases} \frac{dy}{dt} = ky \\ y(0) = P_0 \end{cases}$$

is $y(t) = P_0 e^{kt}$

Def The quantity k is called the proportionality constant or growth rate. P_0 is the initial value.

Def When $P_0, k>0$, the equation $y = P_0 e^{kt}$ is called the exponential growth model

Ex/ An initial pop. of 2 $\leftarrow P_0$ protozoa grows according to

$$\frac{dP}{dt} = \underbrace{0.8}_{k}P$$

where $P(t)$ is the population after t days. Find $P(6)$

$$P(t) = P_0 e^{kt}$$

$$P(t) = 2e^{0.8t}$$

$P(6) = 2e^{0.8(6)} \approx 243$ members

Ex2/ A coloney of E. coli starts w/ an initial pop. of 60 cells, then doubles every 20 min. Find the # of cells after 8 hours

Let t be the time in hours

$$P(t) = P_0 e^{kt}$$

$$P(t) = 60e^{kt}$$

We know: $P(\frac{1}{3}) = 120$, $P(\frac{2}{3}) = 240$, $\boxed{P(1) = 480}$

$$480 = P(1) = 60e^{k\cdot 1}$$
$$480 = 60e^{k}$$
$$8 = e^{k}$$
$$\ln 8 = \ln e^{k}$$
$$k = \ln 8$$

So, $P(t) = 60e^{\ln(8)t}$
$$P(8) = 60e^{\ln(8)\cdot 8} = 60\cdot 2^{24}$$
$$= 1,006,632,960 \text{ cells}$$

(b) When will we reach 20 000 cells?
$$P(t) = 60e^{\ln(8)t}$$
$$20000 = 60e^{\ln(8)t}$$
$$333+\tfrac{1}{3} = e^{\ln(8)t}$$
$$\ln\left(333+\tfrac{1}{3}\right) = \ln(8)t$$
$$t = \frac{\ln(333+\frac{1}{3})}{\ln(8)} \approx 2.7936 \text{ hrs}$$

~~Ex3~~ A pop. has 600 members in 2 years, 6 years later it has 75000 members.

(a) Determine $P(t)$, i.e. the pop. after t years.
$$P(t) = P_0 e^{kt}$$

We know: $P(2) = 600$, $P(8) = 75000$

We need to solve $\begin{cases} 600 = P_0 e^{2k} \\ 75000 = P_0 e^{8k} \end{cases} \longrightarrow P_0 = \dfrac{600}{e^{2k}}$

So,

$$75000 = \frac{600}{e^{2k}} e^{8k}$$

$$75000 = 600 \; e^{6k}$$

$$125 = e^{6k}$$

$$\ln(125) = 6k$$

$$k = \frac{\ln(125)}{6} = \frac{\ln(5^3)}{6} = \frac{3\ln(5)}{6} = \frac{1}{2}\ln(5)$$

Solve for P_0:

$$600 = P_0 e^{2k}$$

$$600 = P_0 e^{2(\frac{1}{2}\ln(5))}$$

$$600 = P_0 e^{\ln(5)}$$

$$600 = 5P_0$$

$$P_0 = 120$$

Thus, $P(t) = 120 e^{\frac{1}{2}\ln(5)t} \approx 120 e^{0.8047t}$

Continuously Compounded Interest

It is common to model the growth of an investment via exponential growth. Let t be the time in years

$$P(t) = P_0 e^{kt}$$

P_0 is called the principal amount and k is the annual interest (as a decimal)

Ex4/ \$3000 is invested at 5% interest

(a) Assuming continuously compounded interest find $P(t)$

$$P(t) = P_0 e^{kt}$$

$$P(t) = 3000 e^{0.05t}$$

(b) Find the amount after

10 years: $P(10) = 3000 e^{0.05(10)} \approx \$4,946.16$

20 years: $P(20) \approx \$8,154.85$

30 years: $P(30) \approx \$13,445.07$

45 years: $P(45) \approx \$28,463.21$

That is a 849% increase.

(c) How long will the investment take to double?

$$2 \cdot 3000 = 3000 e^{0.05t}$$

$$2 = e^{0.05t}$$

$$\ln(2) = 0.05t$$

$$t = \frac{\ln(2)}{0.05} \approx 13.86 \text{ years}$$

Q: How long will it take for an investment to double if it is compounded continuously at $r\%$?

$\frac{r}{100} \leftarrow r\%$ as a decimal

$$P(t) = P_0 e^{\frac{r}{100}t}$$

$$2 \cdot P_0 = P_0 e^{\frac{r}{100}t}$$

$$2 = e^{\frac{r}{100}t}$$

$$\ln(2) = \frac{r}{100}t$$

$$t = \frac{\ln(2) \cdot 100}{r}$$

Theorem (Rule of 72) The amount of time it takes for an investment to double at $r\%$ compounded continuously is

$$t = \frac{\ln(2) \cdot 100}{r} \approx \frac{69}{r} \approx \frac{72}{r} \text{ years}$$

In the previous example

$$t = \frac{\ln(2) \cdot 100}{5} \approx 13.86 \text{ yrs}$$

Rule of 72:

$$t \approx \frac{72}{5} \approx 14.4 \text{ yrs}$$

$$\begin{array}{r} 14 \\ 5\overline{)72} \\ \underline{5} \\ 22 \\ \underline{-20} \\ 2 \end{array}$$

Ex5/ \$1000 is put into a HYSA, after 2 years \$1123.60 remains in the account. Assume it follows exponential growth:

@ Find the amount in the HYSA after t years

$$P(t) = P_0 e^{kt}$$

$$P(t) = 1000 e^{kt}$$

$$1123.60 = 1000e^{k(2)}$$

$$k = \frac{1}{2}\ln(1.12360) \approx \underbrace{0.06}_{6\%}$$

(b) How long will it take for the investment to double?

$$t = \frac{\ln(2)\cdot 100}{6} \approx 11.55 \text{ years}$$

Rule of 72: $t \approx \frac{72}{6} = 12$ yrs