Announcements

1. Exams are on the table
   - Get them; I do not want them
2. Final Exam Review all of next week:
   - Think about what topics/questions you want to see
   - I'll ask Friday
3. EC 4 Due this Sunday @ 11:59PM
4. Complete course evaluations on Brightspace

GOAL: Discuss situations governed by the equation $P(t) = P_0 e^{kt}$ for $k > 0$.

We want to study the differential equation

$$\frac{dy}{dt} = \underset{\uparrow \text{ constant}}{k}\, y$$

To solve, we use separation of variables.

$$\frac{dy}{dt} = ky$$

$$\frac{dy}{dt} = \frac{k}{\frac{1}{y}}$$

$$\frac{1}{y}\, dy = k\, dt$$

$$\int \frac{1}{y}\, dy = \int k\, dt$$

$$\ln|y| + C_1 = kt + C_2$$

$$\ln|y| = kt + [C_2 - C_1]$$

$$|y| = e^{kt + [C_2 - C_1]}$$

$$y = \underbrace{\boxed{\pm e^{C_2 - C_1}}}_{\text{Arbitrary Constant, call it } C} e^{kt}$$

Don't need to know this until MA16020

So the general solution is $y(t) = Ce^{kt}$

Q: What if we add the initial condition $y(0) = P_0$

$$y(t) = Ce^{kt}$$

$$P_0 \overset{\text{set}}{=} y(0) = Ce^{k \cdot 0} = C$$

$$C = P_0$$

Theorem The solution to the IVP

$$\begin{cases} \frac{dy}{dt} = ky \\ y(0) = P_0 \end{cases}$$

is $y(t) = P_0 e^{kt}$

Def The quantity k is called the proportionality constant or the growth rate. P_0 is the initial value.

When P_0, $k > 0$, the equation $y(t) = P_0 e^{kt}$ is called the exponential growth model.

Ex1/ An initial pop. of 2 ← P_0 protozoa grows according to

$$\frac{dP}{dt} = 0.8P \quad \leftarrow k$$

where $P(t)$ is the population after t days. Find $P(6)$.

$$P(t) = P_0 e^{kt}$$

$$P(t) = 2e^{0.8t}$$

$P(6) = 2e^{0.8(6)} \approx 243$ members

Ex2/ A colony of E. coli starts with an initial pop. of 60 cells, then doubles every 20 minutes. Find the # of cells after 8 hours.

$$P(t) = P_0 e^{k(t)} \leftarrow$$

Time in hours

$$P(t) = 60e^{kt}$$

We know: $P(0) = 60$, $P(\frac{1}{3}) = 120$, $P(\frac{2}{3}) = 240$, $P(1) = 480$

$$480 = P(1) = 60e^{k \cdot 1}$$
$$480 = 60e^{k}$$
$$\ln(e^{k}) = \ln(8)$$
$$k = \ln(8)$$

So, $P(t) = 60e^{\ln(8)t}$

$$P(8) = 60 \cdot e^{\ln(8) \cdot 8} = 60 \cdot 2^{24}$$
$$= 1{,}006{,}632{,}960 \text{ cells}$$

(b) When will the population reach 20 000 cells?

$$P(t) = 60e^{\ln(8)t}$$
$$20\,000 = 60e^{\ln(8)t}$$
$$333 + \frac{1}{3} = e^{\ln(8)t}$$
$$\ln(333 + \tfrac{1}{3}) = \ln(8)t$$
$$t = \frac{\ln(333 + \frac{1}{3})}{\ln(8)} \approx 2.7936 \text{ hours}$$

Ex3 A pop. has 600 members after 2 hours, 6 years later it has a pop of 75000.

(a) Assuming exponential growth, find the population after t years.

We know: $P(t) = P_0 e^{kt}$

$P(2) = 600$; $P(8) = 75000$

Need to solve

$$\begin{cases} 600 = P_0 e^{2k} \longrightarrow P_0 = \dfrac{600}{e^{2k}} \\ 75000 = P_0 e^{8k} \end{cases}$$

So,

$$75000 = \frac{600}{e^{2k}} e^{8k}$$

$$75\,000 = 600 e^{6k}$$

$$125 = e^{6k}$$

$$6k = \ln(125)$$

$$k = \frac{\ln(125)}{6} = \frac{\ln(5^3)}{6} = \frac{1}{2}\ln(5)$$

$$P_0 = \frac{600}{e^{2k}} = \frac{600}{e^{2(\frac{1}{2}\ln(5))}} = \frac{600}{e^{\ln(5)}} = \frac{600}{5} = 120$$

So, $P(t) = 120 e^{\frac{1}{2}\ln(5)t} \approx 120 e^{0.8047t}$

Continuously Compounded Interest

It is common to model the growth of an investment via exponential growth. Let t be the time in years

$$P(t) = P_0 e^{kt}$$

P_0 is called the principal amount and k is the annual interest rate (as a decimal).

Ex 4/ \$3000 is invested at 5% interest

ⓐ Assuming continuously compounded interest, find the amount in the account after t years.

$$P(t) = P_0 e^{kt}$$

$$P(t) = 3000\, e^{0.05t}$$

ⓑ Find the amount after

10 years: $P(10) = 3000\, e^{0.05(10)} \approx \$4,946.16$

20 years: $P(20) \approx \$8,154.85$

30 years: $P(30) \approx \$13,445.07$

45 years: $P(45) \approx \$28,463.21$

That is a 849% increase

ⓒ How long will it take for the investment to double?

$$2 \cdot 3000 = 3000\, e^{0.05t}$$

$$2 = e^{0.05t}$$

$$\ln(2) = 0.05t$$

$$t = \frac{\ln(2)}{0.05} \approx 13.86 \text{ years}$$

Q: How long will it take for an investment to double given r% interest compounded continuously?

$\frac{r}{100} \leftarrow$ r% as a decimal

$$P(t) = P_0 e^{\frac{r}{100}t}$$

$$2 \cdot P_0 = P_0 e^{\frac{r}{100}t}$$

$$2 = e^{\frac{r}{100}t}$$

$$\ln(2) = \frac{r}{100}t$$

$$t = \frac{\ln(2) \cdot 100}{r} \text{ years}$$

Theorem (Rule of 72)

The take it takes for an investment to double given r% compounded continuously is:

$$t = \frac{\ln(2) \cdot 100}{r} \approx \frac{69}{r} \approx \frac{72}{r}$$

For the previous ex:

$$t = \frac{\ln(2) \cdot 100}{5} \approx 13.86 \text{ years}$$

Via rule of 72

$$t \approx \frac{72}{5} = 14.4 \text{ yrs}$$

$$\begin{array}{r} 14 \\ 5\overline{)72} \\ -5 \\ 22 \\ -20 \\ 2 \end{array}$$

Ex 5) \$1000 is put into a HYSA, after 2 years \$1123.60 remains in the account. Assume cont. comp. int.

ⓐ Find the amount in the HYSA after t yrs

$$P(t) = P_0 e^{kt}$$

$$P(t) = 1000 e^{kt}$$

$$1123.60 = 1000 e^{2k}$$

$$1.12360 = e^{2k}$$

$$\ln(1.12360) = 2k$$

$$k = \frac{1}{2}\ln(1.12360) \approx \underbrace{0.06}_{6\% \text{ interest}}$$

So, $P(t) = 1000 e^{0.06t}$

ⓑ How long will it take for it to double?

$$t = \frac{\ln(2) \cdot 100}{6} \approx 11.55 \text{ years}$$

Rule of 72:

$$t \approx \frac{72}{6} = 12 \text{ years}$$