GOAL: Discuss situations governed by the equation $P(t) = P_0 e^{kt}$ for $k < 0$.

Recall that the solution to the diff eq

$$\frac{dy}{dt} = ky$$

takes the form $y(t) = Ce^{kt}$

When $k > 0$, $C > 0$, $y \to \infty$ as $t \to \infty$

When $k < 0$, $C > 0$, $y \to 0$ as $t \to \infty$

We call this situation <u>exponential decay</u>.

Ex/ Solve the IVP $\begin{cases} \frac{dy}{dt} = -\frac{1}{2}y \\ y(2) = \frac{3}{e} \end{cases}$ k

$$y(t) = Ce^{kt}$$

$$\frac{3}{e} \overset{set}{=} y(0) = Ce^{k \cdot 2} = Ce^{k(2)} = Ce^{-\frac{1}{2}(2)} = Ce^{-1}$$

$$\frac{3}{e} = \frac{C}{e}$$

$$C = 3$$

So, $y(t) = 3e^{-\frac{1}{2}t}$

<u>Half Life</u>

The decay of radioactive isotopes follows an exponential decay model.

<u>Def</u> The half-life of a substance is the time required for 50% of a sample to decay.

Ex2/ Bismuth - 210 ($^{210}_{83}Bi$) has a half life of 5 days. Suppose we have a sample of 800 mg

ⓐ Find the mass remaining after 30 days

$$P(t) = P_0 e^{kt}$$
$$P(t) = 800 e^{kt}$$
$$\tfrac{1}{2} \cdot 800 = 800 e^{5k}$$
$$\tfrac{1}{2} = e^{5k}$$
$$\ln(1/2) = 5k \qquad \tfrac{1}{2} = 2^{-1}$$
$$k = \frac{\ln(1/2)}{5} = -\frac{\ln(2)}{5}$$

So, $P(t) = 800 e^{-\frac{\ln(2)}{5} t}$

$$P(30) = 800 e^{-\frac{\ln(2)}{5} \cdot 30} = \frac{800}{2^6}$$
$$= 12.5 \text{ mg}$$

ⓑ How long will it take so only 1 mg remains?

$$P(t) = 800 e^{-\frac{\ln(2)}{5} t}$$
$$1 = 800 e^{-\frac{\ln(2)}{5} t}$$
$$800^{-1}$$
$$\ln\left(\frac{1}{800}\right) = \ln\left(e^{-\frac{\ln(2)}{5} t}\right)$$
$$-\ln(800) = -\frac{\ln(2)}{5} t$$
$$t = \frac{5 \cdot \ln(800)}{\ln(2)} \approx 48.22 \text{ days}$$

Ex 2/ After 3 days, 58% of a sample of Radon-222 ($^{222}_{86}Ra$) remains

ⓐ Determine the half life of $^{222}_{86}Ra$?

$$P(t) = 100e^{kt} \qquad 0.58(P_0) = P_0 e^{kt}$$

$$58 = 100e^{3k}$$

$$\frac{58}{100} = e^{3k}$$

$$\ln(58/100) = 3k$$

$$k = \frac{\ln(58/100)}{3}$$

Q: How is k and half life related?

Let h denote the half life

$$\frac{1}{2}P_0 = P_0 e^{kh}$$

$$\frac{1}{2} = e^{kh}$$

$$\ln\left(\frac{1}{2}\right) = kh$$

$$k = \frac{\ln(1/2)}{h} = -\frac{\ln(2)}{h} \longleftrightarrow h = -\frac{\ln(2)}{k}$$

Back to the example

$$\text{Half-Life} = -\frac{\ln(2)}{\frac{\ln(58/100)}{3}} \approx 3.82 \text{ days}$$

ⓑ How long will take for the sample to decay to 10% the original amount?

$$P(t) = P_0\, e^{\frac{\ln(58/100)}{3} t}$$

$$0.1 P_0 = P_0\, e^{\frac{\ln(58/100)}{3} t}$$

$\frac{1}{10} = 10^{-1}$

$$0.1 = e^{\frac{\ln(58/100)}{3} t}$$

$$-\ln(10) = \frac{1}{3}\ln\left(\frac{58}{100}\right)t$$

$$t = -\frac{\ln(10)\cdot 3}{\ln(58/100)} \approx 12.68 \text{ days}$$

Carbon Dating

The half life of Carbon-14 $\left({}^{14}_{6}C\right)$ is roughly 5715 years. [5700 ± 30]

Ex3/ A parchment fragment was discovered to have 74% of the amount of ${}^{14}_{6}C$ as present day plant material. Estimate the age of the parchment.

$$50 = 100\, e^{5715 k}$$

$$\frac{1}{2} = e^{5715 k}$$

$$-\ln(2) = 5715 k$$

$$k = -\frac{\ln(2)}{5715} \approx -0.000121$$

So, $P(t) = P_0\, e^{-\frac{\ln(2)}{5715} t}$

(ii) Estimate age

$$74 = 100 e^{-\frac{\ln(2)}{5715}t}$$

$$\frac{74}{100} = e^{-\frac{\ln(2)}{5715}t}$$

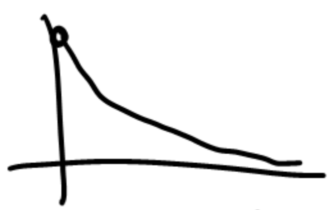
$$\ln(74/100) = \ln e^{-\frac{\ln(2)}{5715}t}$$

$$\ln(74/100) = -\frac{\ln(2)}{5715}t$$

$$t = \frac{\ln(74/100)}{-\frac{\ln(2)}{5715}} \approx 2482.6 \text{ years}$$

Drugs Leaving the Body

Def If a drug leaves your system at a constant rate, it is called a 0-order elimination drug. If it follows exp. decay, it is a 1st-order elim. drug.



Ex4 At low doses, caffeine is a 1st-order elim. drug with a "half-life" of 5 hrs. If someone consumes 2 cups of coffee (~180mg) at 7AM, what percentage remains at 3 PM?

$$90 = 180\, e^{5k}$$
$$k = -\frac{\ln(2)}{5}$$

So, $P(t) = 180\, e^{-\frac{\ln(2)}{5} t}$

Ie, find $P(8) = 180\, e^{-\frac{\ln(2)}{5}(8)} \approx 59.38$ mg

Percentage: $\left[\frac{P(8)}{P(0)}\right] \cdot 100 = \frac{59.38}{180} \cdot 100 \approx 33\%$

Non Examinable

Newton's Law of Cooling: The rate an object cools is governed by the IVP

$$\begin{cases} \frac{dT}{dt} = k(T - T_a) \\ T(0) = T_0 \end{cases}$$

where $T(t)$ is the temp. and T_a is the ambient temp.

Let $y(t) = T(t) - T_a$

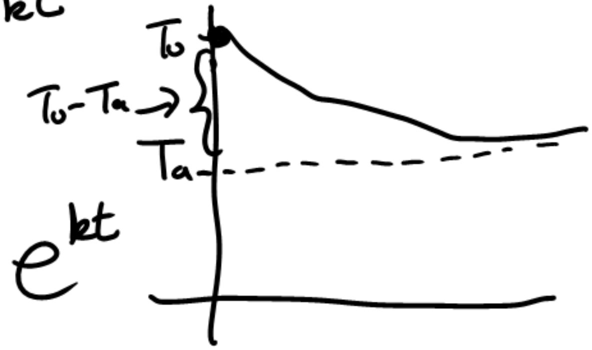
$$\frac{dy}{dt} = \frac{dT}{dt}$$

$$\frac{dT}{dt} = k(T - T_a) \longrightarrow \frac{dy}{dt} = ky$$

$$y(t) = Ce^{kt}$$

$$T(t) - T_a = Ce^{kt}$$

$$T(t) = T_a + Ce^{kt}$$



$\underline{T(0) = T_0}$

$$T(t) = T_a + [T_0 - T_a]e^{kt}$$

Ex5/ A ham is taken out of the oven and is placed into a 75°F room. The initial temp of the ham is 185°F

@ $\frac{1}{2}$ hr later, the temp is 150°F. What is the temperature after 45 mins.

$T_a = 75$

$T_0 = 185$

$$\begin{array}{r} 185 \\ \underline{75} \\ 110 \end{array}$$

So,
$$T(t) = 75 + 110e^{kt}$$
$$150 = 75 + 110e^{30k}$$
$$\frac{75}{110} = e^{30k}$$
$$\ln(75/110) = 30k$$
$$k = \frac{\ln(75/110)}{30}$$

So,
$$T(t) = 75 + 110e^{\frac{\ln(75/110)}{30}t}$$
$$T(45) \approx 137°F$$