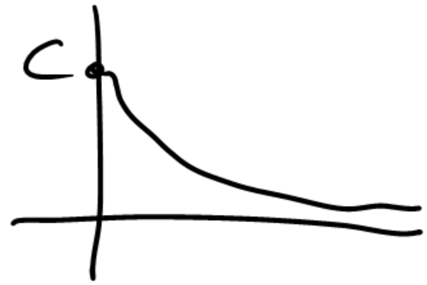
GOAL: Discuss situations governed by the equation $P(t) = P_0 e^{kt}$ for $k < 0$.

Recall that a solution to the diff eq

$$\frac{dy}{dt} = ky$$



takes the form $y(t) = Ce^{kt}$

when $k>0$, $C>0$, $y \to \infty$ as $t \to \infty$

when $k<0$, $C>0$, $y \to 0$ as $t \to \infty$

This situation is called exponential decay.

Ex/ Solve the IVP $\begin{cases} \frac{dy}{dt} = -\frac{1}{2}y \\ y(2) = \frac{3}{e} \end{cases}$ $\leftarrow k$

$$y(t) = Ce^{kt}$$

$$y(t) = Ce^{-\frac{1}{2}t}$$

$$\frac{3}{e} \overset{\text{set}}{=} y(2) = Ce^{-\frac{1}{2}(2)} = Ce^{-1}$$

$$\frac{3}{e} = \frac{C}{e}$$

$$C = 3$$

So, $P(t) = 3e^{-\frac{1}{2}t}$

<u>Half-Life</u>

The decay of radioactive isotopes follows an exponential decay model.

<u>Def</u> The half-life of a substance is the time required for 50% of the sample to decay.

Ex 2/ Bismuth-210 ($^{210}_{83}Bi$) has a half life of 5 days.
Suppose we have a sample size of 800mg
ⓐ Find the mass remaining after 30 days

$$P_{(t)} = P_0 e^{kt}$$

$$P(t) = 800 e^{kt}$$

$$\frac{1}{2} \cdot 800 = 800 e^{k(5)}$$

$$\frac{1}{2} = e^{5k}$$

2^{-1}

$$\ln(1/2) = 5k$$

$$-\ln(2) = 5k$$

$$k = -\frac{\ln(2)}{5}$$

So, $P(t) = 800 e^{-\frac{\ln(2)}{5} t}$

$$P(30) = 800 e^{-\frac{\ln(2)}{5} \cdot 30} = \frac{800}{2^6} = 12.5 \text{ mg}$$

ⓑ How long until only 1 mg remains?

$$P(t) = 800 e^{-\frac{\ln(2)}{5} t}$$

$$1 = 800 e^{-\frac{\ln(2)}{5} t}$$

$$\frac{1}{800} = e^{-\frac{\ln(2)}{5} t}$$

$$-\ln(800) = -\frac{\ln(2)}{5} t$$

$$t = \frac{\ln(800) \cdot 5}{\ln(2)} \approx 48.22 \text{ days}$$

Ex3/ After 3 days 58% of a sample of Radon - 222 ($^{222}_{86}Ra$) remains

(a) Determine the half-life

$$0.58\,P_0 = P_0 e^{kt}$$

$$P(t) = P_0 e^{kt}$$

$$58 = 100 e^{k(3)}$$

$$58 = 100 e^{3k}$$

$$\frac{58}{100} = e^{3k}$$

$$\ln(58/100) = 3k$$

$$k = \frac{\ln(58/100)}{3}$$

Q: How are k and the half life related?

Let h denote the half life

$$\frac{1}{2} \cdot P_0 = P_0 e^{kh}$$

$$\frac{1}{2} = e^{kh}$$

$$\ln(\tfrac{1}{2}) = kh$$

$$-\ln(2) = kh$$

$$k = \frac{-\ln(2)}{h} \longleftrightarrow h = \frac{-\ln(2)}{k}$$

Back to the ex

$$\text{Half-Life} = \frac{-\ln(2)}{k} = \frac{-\ln(2)}{\frac{\ln(58/100)}{3}} \approx 3.82 \text{ days}$$

(b) How long will it take for the sample to decay to 10% the original amount

$$P(t) = P_0 e^{\frac{\ln(58/100)}{3} t}$$

$$10 = 100 e^{\frac{\ln(58/100)}{3} t}$$

$10^{-1} \rightarrow$ $$\frac{1}{10} = e^{\frac{\ln(58/100)}{3} t}$$

$$-\ln(10) = \frac{1}{3} \ln\left(\frac{58}{100}\right) t$$

$$t = \frac{-\ln(10) \cdot 3}{\ln\left(\frac{58}{100}\right)} \approx 12.68 \text{ days}$$

Carbon-Dating

The half-life of Carbon-14 ($^{14}_{6}C$) is roughly 5715 years. [5700 ± 30]

Ex3/ A parchment fragment was discovered to have 74% of the amount of $^{14}_{6}C$ as present day plant matter. Estimate the age of the parchment.

(i) Determine k:

$$50 = 100 e^{5715 k}$$

$$k = \frac{-\ln(2)}{5715} \approx -0.000121$$

(ii) Estimate age

$$P(t) = P_0 e^{\frac{-\ln(2)}{5715} t}$$

$$74 = 100 e^{\frac{-\ln(2)}{5715} t}$$

$$\frac{74}{100} = e^{-\frac{\ln(2)}{5715} t}$$

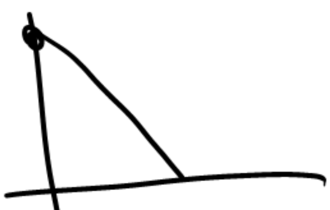
$$\ln\left(\frac{74}{100}\right) = -\frac{\ln(2)}{5715} t$$

$$t = \frac{\ln(74/100)}{-\frac{\ln(2)}{5715}} \approx 2482.6 \text{ years}$$

Drugs Leaving the Body

Def If a drug leaves your system at a constant rate, it is called a 0-order elimination drug.

If it follows exp. decay, it is a 1^{st}-order elim. drug.



Ethanol

Ibuprofen

~~Ex4.~~ At low doses caffeine is a 1^{st} order elim. drug with a "half-life" of 5 hours. If someone consumes 2 cups of coffee (~180mg) at 7AM, what percentage will be in their system at 3PM?

$$P(t) = P_0 e^{kt}$$

$$90 = 180 e^{5k}$$

$$k = -\frac{\ln(2)}{5}$$

So, $P(t) = 180 e^{-\frac{\ln(2)}{5} t}$

$$P(8) = 180 e^{-\frac{\ln(2)}{5} \cdot 8} \approx 59.38 \text{ mg}$$

$$\text{Percentage} = \left[\frac{P(8)}{P(0)}\right] \cdot 100 = \frac{59.38}{180} \cdot 100 \approx 33\%$$

Non-Examinable

Newton's Law of Cooling: If $T(t)$ is the temperature of an object and T_a is the ambient temp. The rate the object cools is governed by

$$\begin{cases} \frac{dT}{dt} = k(T(t) - T_a) \\ T(0) = T_0 \end{cases}$$

Let $y(t) = T(t) - T_a$

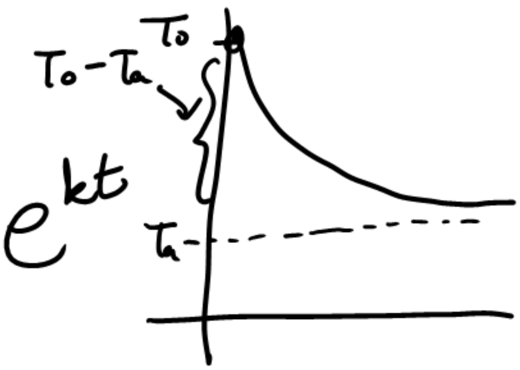
$$\frac{dy}{dt} = \frac{dT}{dt}$$

$$\frac{dT}{dt} = k(T(t) - T_a) \longrightarrow \frac{dy}{dt} = ky$$

$$y(t) = Ce^{kt}$$

$$T(t) - T_a = Ce^{kt}$$

$$T(t) = T_a + Ce^{kt}$$



$T(0) = T_0$

$$T(t) = T_a + [T_0 - T_a]e^{kt}$$

Ex5 A ham is taken out of the oven and is placed into a 75°F room. The initial temp of the ham is 185°F

@ $\frac{1}{2}$ hr later, the temp is 150°F. What is the temperature after 45 mins.

$T_a = 75$
$T_0 = 185$

$$\begin{array}{r} 185 \\ -75 \\ \hline 110 \end{array}$$

$$T(t) = 75 + 110\,e^{kt}$$

$$150 = 75 + 110\,e^{30k}$$

$$\frac{75}{110} = e^{30k}$$

$$\ln\left(75/110\right) = 30k$$

$$k = \frac{\ln(75/110)}{30}$$

So,
$$T(t) = 75 + 110\,e^{\frac{\ln(75/110)}{30}t}$$

$$T(45) \approx 137°F$$