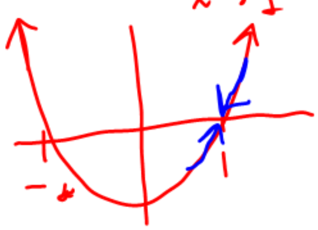
**GOAL:** Compute limits without guessing.

We start with non-piecewise functions.

To find $\lim_{x\to a} f(x)$:

**Case 1:** $f(a)$ is well-defined [not undefined, not $\infty$, not $-\infty$]

Ex1 $\lim_{x\to 1}(x^2+x-2) = 1^2+1-2 = 0$



In this case, we say $f(x)$ is <u>continuous</u> at $x=1$.

$$\lim_{x\to a} f(x) = f\left(\lim_{x\to a} x\right) = f(a)$$

**Case 2:** $f(a)$ takes the form $\frac{[\text{Non-zero}]}{0}$. Then the limit is $\infty$, $-\infty$, or DNE.

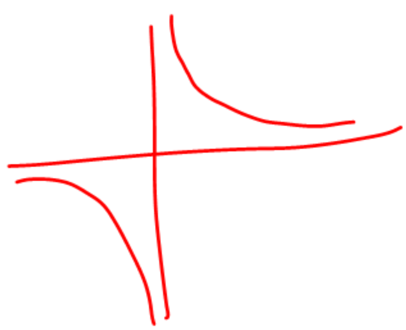
Ex2 $\lim_{x\to 0}\frac{1}{2x}$

$$\lim_{x\to 0^-}\frac{1}{2x} = -\infty$$

$$\lim_{x\to 0^+}\frac{1}{2x} = \infty$$

| -0.1 | -0.01 | -0.001 |
|---|---|---|
| -100 | -10000 | -1000000 |

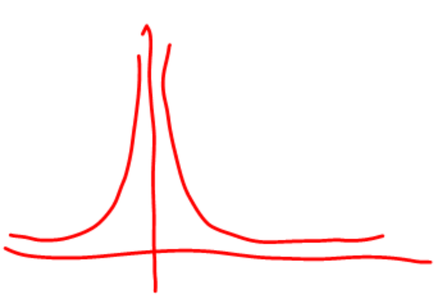
| | $x<0$ | 0 | $x>0$ |
|---|---|---|---|
| Top | + | | + |
| Bottom | − | | + |
| Frac | − | | + |



Ex3 $\lim_{x\to 0}\frac{1}{x^2}$

$$\lim_{x\to 0^-}\frac{1}{x^2} = \infty$$

$$\lim_{x\to 0^+}\frac{1}{\underbrace{x^2}_{\geq 0}} = \infty$$

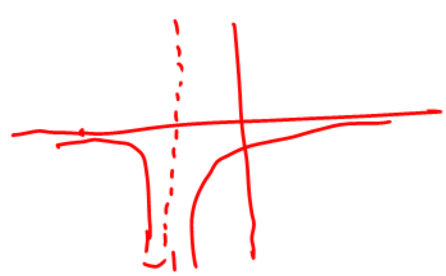


$$\lim_{x\to 0}\frac{1}{2x} = \infty$$

$\lim_{x\to -1}\frac{-5}{(x+1)^2}$

$$\lim_{x\to -1^-}\frac{-5}{(x+1)^2} = -\infty$$

$$\lim_{x\to -1^+}\frac{\overbrace{-5}^{\leq 0}}{\underbrace{(x+1)^2}_{\geq 0}} = -\infty$$



**Case 3** $f(a)$ takes an indeterminant form $\left(\frac{0}{0}\right)$

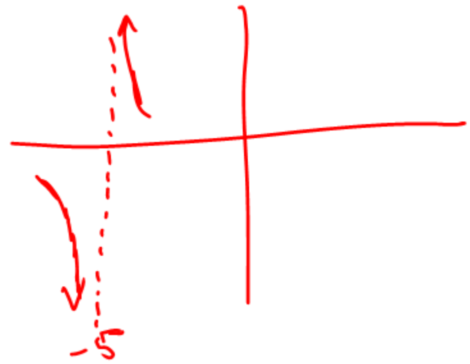
**Ex 4**
$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)} = \lim_{x\to 1} (x^2+x+1) = 1+1+1 = 3$$

Difference of Cubes: $a^3-b^3=(a-b)(a^2+ab+b^2)$

**Goal** Use algebra to reduce the limit to Case 1 or 2.

**Ex 6**
$$\lim_{x\to 5} \frac{x^3-5x^2}{(x-5)^2} = \lim_{x\to 5} \frac{x^2(x-5)}{(x-5)^2} = \lim_{x\to 5} \frac{x^2}{x-5} \text{ DNE}$$

| | $x<5$ | $5$ | $x>5$ |
|---|---|---|---|
| Top | + | | + |
| Bottom | − | | + |
| Frac | − | | + |

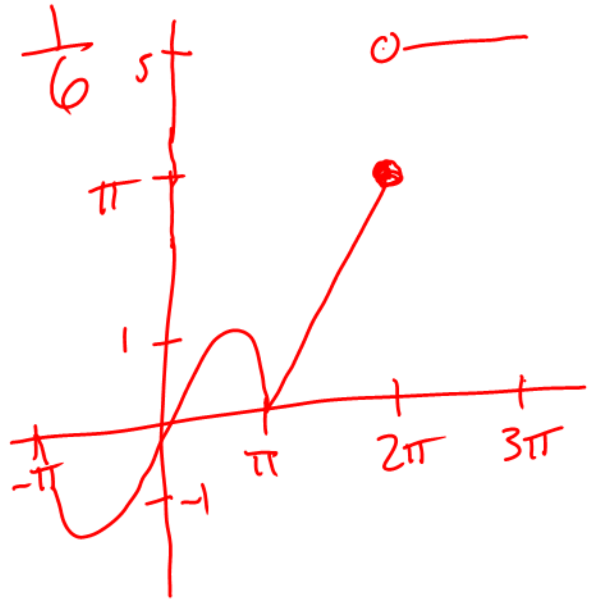



**Ex 7**
$$\lim_{x\to 2} \frac{x^2-3x+2}{x^2+2x-8} = \lim_{x\to 2} \frac{(x-1)\cancel{(x-2)}}{(x+4)\cancel{(x-2)}} = \lim_{x\to 2} \frac{x-1}{x+4}$$
$$= \frac{2-1}{2+4} = \frac{1}{6}$$

**Piecewise Functions**

**Ex 8** Suppose

$$f(x)=\begin{cases} \sin x & \text{if } -\pi \le x \le \pi \\ x-\pi & \text{if } \pi < x \le 2\pi \\ 5 & \text{if } 2\pi < x \le 3\pi \end{cases}$$



Case 1 $x=a$ is not a "boundary point"

(Boundary pts $x=\pi, 2\pi$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \sin x = \sin\frac{\pi}{2} = 1$$

Case 2 $x=a$ is a boundary point

$$\lim_{x\to 2\pi} f(x) \begin{cases} \lim_{x\to 2\pi^-} f(x) = \lim_{x\to 2\pi^-} (x-\pi) = 2\pi-\pi=\pi \\ \lim_{x\to 2\pi^+} f(x) = \lim_{x\to 2\pi^+} 5 = 5 \end{cases}$$

So $\lim_{x\to 2\pi} f(x)$ DNE

Limit Laws

Loosely speaking, if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists.
The limit "plays well" with arithmetic.

- $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x)\, g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
- $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ ← Needs to be not 0

In particular,

$\lim_{x\to a}(k\cdot f(x)) = k \lim_{x\to a} f(x)$ ; $k$ is a constant

$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ for $n=1,2,3,\ldots$

Ex

If $\lim_{x\to 3} f(x) = 5$ and $\lim_{x\to 3} g(x) = 2$. Then

$$\lim_{x\to 3} [f(x) + g(x) + 3]^2 = \left[\lim_{x\to 3} (f(x) + g(x) + 3)\right]^2$$

$$= \left[\lim_{x\to 3} f(x) + \lim_{x\to 3} g(x) + \lim_{x\to 3} 3\right]^2$$

$$= [5 + 2 + 3]^2 = 10^2 = 100$$

NOTE: The limits need to exist to apply the limit laws.

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x+x^2}\right) = \lim_{x\to 0} \frac{x + x^2 - x}{x(x+x^2)} = \lim_{x\to 0} \frac{x^2}{x^2(x+1)}$$

$$= 1$$