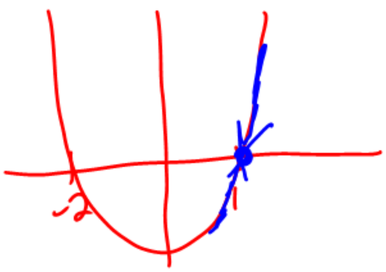
GOAL: Compute limits without guessing.

We start with non-piecewise functions.

To find $\lim_{x\to a} f(x)$:

Case 1: $f(a)$ is well-defined [not undefined, not ∞, not $-\infty$]

Ex1 $\lim_{x\to 1}(x^2+x-2) = 1^2+1-2 = 0$



In this example, we say $f(x)$ is Continuous at $x=1$. In general,

$$\lim_{x\to a} f(x) = f\left(\lim_{x\to a} x\right) = f(a)$$

Case 2: $f(a)$ takes the form $\frac{[\text{Non-Zero}]}{0}$. Then the limit is either ∞, $-\infty$, or DNE.

Ex2 $\lim_{x\to 0}\frac{1}{2x}$

$\lim_{x\to 0^-}\frac{\overset{>0}{1}}{\underset{<0}{2x}} = -\infty$

$\lim_{x\to 0^+}\frac{\overset{>0}{1}}{\underset{>0}{2x}} = \infty$

	$x<0$	0	$x>0$
Top	+		+
Bottom	−		+
Frac	−		+

Ex3 $\lim_{x\to 0}\frac{1}{x^2}$

$\lim_{x\to 0^-}\frac{1}{x^2} = \infty$

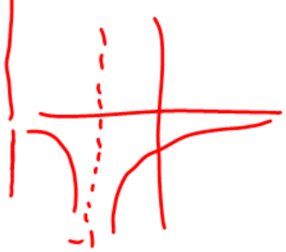
$\lim_{x\to 0^+}\frac{1}{x^2} = \infty$

	$x<0$	0	$x>0$
Top	+		+
Bottom	+		+

$\lim_{x\to -1}\frac{-5}{(x+1)^2}$

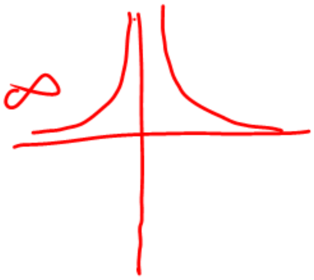
$\lim_{x\to -1^-}\frac{-5}{(x+1)^2} = -\infty$

$\lim_{x\to -1^+}\frac{-5}{(x+1)^2} = -\infty$



$\lim_{x\to -1}\frac{-5}{(x+1)^2} = -\infty$

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$



	$x<-1$	-1	$x>-1$
Top	$-$		$-$
Bottom	$+$		$+$

Case 3 $f(a)$ gets us an indeterminate form $\left(\frac{0}{0}\right)$

Ex 4

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} = \lim_{x\to 1}(x^2+x+1) = 1+1+1 = 3$$

Difference of Cubes: $a^3-b^3=(a-b)(a^2+ab+b^2)$

Goal Use Algebra to reduce the limit to Case 1 or case 2.

Ex 5

$$\lim_{x\to 5} \frac{x^3-5x^2}{(x-5)^2} = \lim_{x\to 5} \frac{x^2(x-5)}{(x-5)^2} = \lim_{x\to 5} \frac{x^2}{x-5} \quad \text{DNE}$$

$-0.1 = 4.9-5 < 0$

$5.1-5 = 0.1 > 0$

	$x<5$	5	$x>5$
Top	$+$		$+$
Bottom	$-$		$+$
	$-\infty$		∞

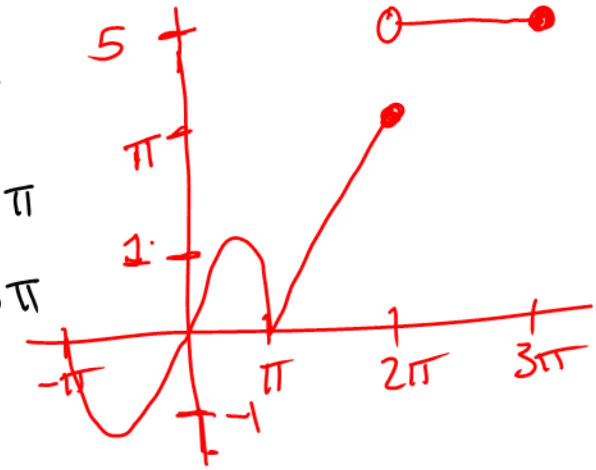
Ex 6

$$\lim_{x\to 2} \frac{x^2-3x+2}{x^2+2x-8} = \lim_{x\to 2} \frac{(x-1)\cancel{(x-2)}}{(x+4)\cancel{(x-2)}} = \lim_{x\to 2} \frac{x-1}{x+4}$$

$$= \frac{2-1}{2+4} = \frac{1}{6}$$

Piecewise Functions

Suppose
$$f(x)=\begin{cases} \sin x & \text{if } -\pi \le x \le \pi \\ x-\pi & \text{if } \pi < x \le 2\pi \\ 5 & \text{if } 2\pi < x \le 3\pi \end{cases}$$



Case 1 Our point of interest is not a "boundary point" (bdy. pts in the example are π and 2π).

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \sin x = \sin\frac{\pi}{2} = 1$$

Case 2 The point is a boundary point

$$\lim_{x\to 2\pi} f(x) \quad \begin{cases} \lim_{x\to 2\pi^-} f(x) = \lim_{x\to 2\pi^-}(x-\pi) = 2\pi-\pi = \pi \\ \lim_{x\to 2\pi^+} f(x) = \lim_{x\to 2\pi^+} 5 = 5 \end{cases}$$

$$\lim_{x\to 2\pi} f(x) \text{ DNE}$$

Limit Laws

Loosely speaking if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists, then the limits "play well" with arithmetic.

- $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x)][g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$

- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \leftarrow \neq 0$

In particular,

- $\lim_{x\to a} k\cdot f(x) = k \lim_{x\to a} f(x)$ for constant k
- $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$ for $n = 1, 2, 3, \ldots$

Ex

If $\lim_{x\to 3} f(x) = 5$ and $\lim_{x\to 3} g(x) = 2$, then

$$\lim_{x\to 3} [f(x) + g(x) + 3]^2 = [\lim_{x\to 3} (f(x) + g(x) + 3)]^2$$

$$= [\lim_{x\to 3} f(x) + \lim_{x\to 3} g(x) + \lim_{x\to 3} 3]^2$$

$$= [5 + 2 + 3]^2 = 10^2 = 100$$

Remark The limits need to exist to apply the Limit Laws.

Ex/

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x+x^2}\right) = \lim_{x\to 0} \frac{x+x^2-x}{x(x+x^2)} = \lim_{x\to 0} \frac{x^2}{x(x+x^2)}$$

$$= \lim_{x\to 0} \frac{x^2}{x^2(x+1)} = \lim_{x\to 0} \frac{1}{x+1} = 1$$