Lecture 3 Finding Limits Analytically

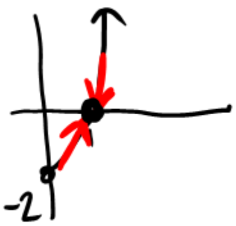
#7 HW2 $\lim_{x\to 0}\frac{ke^x-k}{x}=k$

Goal Compute Limits without guess

For a non-piecewise function $f(x)$, for $\lim_{x\to a} f(x)$:

Case 1: $f(a)$ is well-defined (not undefined, not ∞, not $-\infty$)

Ex/ Compute $\lim_{x\to 1}(x^2+x-2) = 1^2+1-2=0$



In this case, f is said to be continuous at $x=1$. In general,

$$\lim_{x\to a} f(x) = f(a) = f\left(\lim_{x\to a} x\right)$$

Case 2: $f(a)$ takes the form $\frac{[\text{non-zero number}]}{0}$. Then $\lim_{x\to a} f(x)$ is either ∞, $-\infty$, or DNE.

Ex/ $\lim_{x\to 0}\frac{1}{x^2}$ $\begin{cases}\lim_{x\to 0^-}\frac{1}{x^2} \\ \lim_{x\to 0^+}\frac{1}{x^2}\end{cases} = \infty$

Always Positive

Ex/ $\lim_{x\to -1}\frac{-5}{(x+1)^2}$ $\begin{cases}\lim_{x\to -1^-}\frac{-5}{(x+1)^2} = -\infty \\ \lim_{x\to -1^+}\frac{-5}{(x+1)^2} = -\infty\end{cases}$

Always Positive

Ex/ Evaluate $\lim_{x\to 0}\frac{1}{2x}$ $\begin{cases}\lim_{x\to 0^-}\frac{1}{2x} = -\infty & \text{Negative} \\ \lim_{x\to 0^+}\frac{1}{2x} = \infty & \text{Positive}\end{cases}$ $\lim_{x\to 0}\frac{1}{2x}$ DNE

Case 3: $f(a)$ takes an indeterminate form (like $\frac{0}{0}$)

Ex/ $\lim_{x\to 1}\frac{x^3-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{(x-1)} = \lim_{x\to 1}(x^2+x+1) = 1^2+1+1 = 3$

Goal To compute $\lim_{x\to a} f(x)$, use algebra to reduce the limit to Case 1 or 2.

Ex/ $\lim\limits_{x\to 5} \dfrac{x^3-5x^2}{(x-5)^2} = \lim\limits_{x\to 5} \dfrac{x^2(x-5)}{(x-5)^2} = \lim\limits_{x\to 5^-} \dfrac{x^2}{x-5}$ $\left(\dfrac{25}{0}\right)$

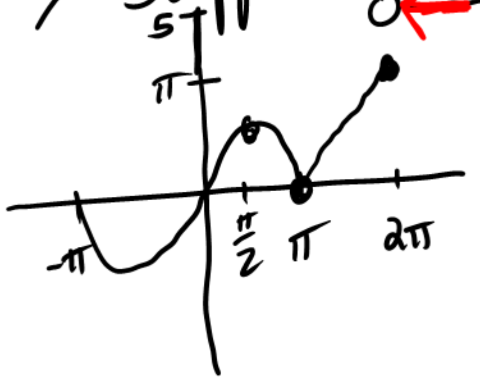
$x<5$	5	$x>5$
$\dfrac{+}{-}$		$\dfrac{+}{+}$

$\lim\limits_{x\to 5} \dfrac{x^2}{x-5}$ DNE

Ex/ $\lim\limits_{x\to 2} \dfrac{x^2-3x+2}{x^2+2x-8} = \lim\limits_{x\to 2} \dfrac{(x-1)\cancel{(x-2)}}{(x+4)\cancel{(x-2)}} = \lim\limits_{x\to 2} \dfrac{x-1}{x+4} = \dfrac{2-1}{2+4} = \dfrac{1}{6}$

Piecewise Functions

Ex/ Suppose $f(x) = \begin{cases} \sin x & \text{if } -\pi \le x \le \pi \\ x-\pi & \text{if } \pi < x \le 2\pi \\ 5 & \text{if } 2\pi < x \end{cases}$



To compute $\lim\limits_{x\to a} f(x)$

In this ex. π and 2π

Case 1 $x=a$ is not a "boundary point"

Ex/ $\lim\limits_{x\to \frac{\pi}{2}} f(x) = \lim\limits_{x\to\frac{\pi}{2}} \sin x = \sin\frac{\pi}{2} = 1$

Case 2 $x=a$ is a "boundary point"

Ex/ $\lim\limits_{x\to\pi} f(x)$

$\lim\limits_{x\to\pi^-} f(x) = \lim\limits_{x\to\pi^-} \sin x = \sin\pi = 0$

$\lim\limits_{x\to\pi^+} f(x) = \lim\limits_{x\to\pi^+} (x-\pi) = \pi-\pi = 0$

So $\lim\limits_{x\to\pi} f(x) = 0$

Ex/ $\lim\limits_{x\to 2\pi} f(x)$

$\lim\limits_{x\to 2\pi^-} f(x) = \lim\limits_{x\to 2\pi^-} (x-\pi) = 2\pi-\pi = \pi$

$\lim\limits_{x\to 2\pi^+} f(x) = \lim\limits_{x\to 2\pi^+} (5) = 5$

So, $\lim\limits_{x\to 2\pi} f(x)$ DNE

Theorem (Limit Laws) Suppose c, k, L, and G be real numbers. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = G$. Then

- $\lim_{x\to c}(kf(x)) = k\left(\lim_{x\to c} f(x)\right) = kL$
- $\lim_{x\to c}(f(x) \pm g(x)) = \left(\lim_{x\to c} f(x)\right) \pm \left(\lim_{x\to c} g(x)\right) = L \pm G$
- $\lim_{x\to c}(f(x)\,g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right) = LG$
- $\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{G}$ (provided $G \neq 0$)
- $\lim_{x\to c}[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n = L^n$ for $n = 1, 2, 3, \dots$

<u>NOTE:</u> The limit laws only holds when the limits of f and g exist.

Ex/ $\lim_{x\to 0}\left(\frac{1}{x}\right)\sin x = 1$; however, $\lim_{x\to 0}\frac{1}{x}$ DNE. So

$$\lim_{x\to 0}\left(\frac{1}{x}\right)\sin x \neq \left(\lim_{x\to 0}\frac{1}{x}\right)\left(\lim_{x\to 0}\sin x\right)$$

Ex/ If $\lim_{x\to 3} f(x) = 5$, and $\lim_{x\to 3} g(x) = 2$, then

$$\lim_{x\to 3}[f(x)+g(x)]^2 = \left[\lim_{x\to 3}(f(x)+g(x))\right]^2 = \left[\left(\lim_{x\to 3} f(x)\right) + \left(\lim_{x\to 3} g(x)\right)\right]^2$$
$$= [5+2]^2 = 7^2 = 49$$

Ex/ Compute $\lim_{x\to 0}\frac{(\cos x - 1)^2 \sin x}{x^3}$

$$= \lim_{x\to 0}\left(\frac{\cos x - 1}{x}\right)^2 \cdot \frac{\sin x}{x} = \Big[\underbrace{\lim_{x\to 0}\frac{\cos x - 1}{x}}_{0}\Big]^2 \cdot \Big[\underbrace{\lim_{x\to 0}\frac{\sin x}{x}}_{1}\Big] = 0$$