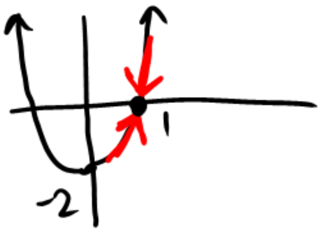
Lecture 3 Finding Limits Analytically

Goal Compute Limits without guessing

For a non-piecewise function $f(x)$, to compute $\lim_{x \to a} f(x)$

Case 1: $f(a)$ is well-defined (not undefined, not ∞, not $-\infty$)

Ex/ $\lim_{x \to 1} (x^2 + x - 2) = 1^2 + 1 - 2 = 0$



In this case, we say $f(x)$ is <u>continuous</u> at $x = 1$. In general,

$$\lim_{x \to a} f(x) = f(a) = f\left(\lim_{x \to a} x\right)$$

Case 2: $f(a)$ takes the form $\frac{[\text{non-zero number}]}{0}$. Then $\lim_{x \to a} f(x)$ is either ∞, $-\infty$, or DNE.

Ex/ $\lim_{x \to 0} \frac{1}{x^2}$

$$\lim_{x \to 0^-} \frac{1}{x^2} = \infty$$

$$\lim_{x \to 0^+} \frac{1}{x^2} = \infty$$

Always Positive

So $\lim_{x \to 0} \frac{1}{x^2} = \infty$

$\lim_{x \to -1} \frac{-5}{(x+1)^2}$

$$\lim_{x \to -1^-} \frac{-5}{(x+1)^2} = -\infty$$

$$\lim_{x \to -1^+} \frac{-5}{(x+1)^2} = -\infty$$

Always Positive

So $\lim_{x \to -1} \frac{-5}{(x+1)^2} = -\infty$

Ex/ $\lim_{x \to 0} \frac{1}{2x}$

$$\lim_{x \to 0^-} \frac{1}{2x} = -\infty \qquad (1 > 0,\ 2x < 0)$$

$$\lim_{x \to 0^+} \frac{1}{2x} = \infty \qquad (1 > 0,\ 2x > 0)$$

So $\lim_{x \to 0} \frac{1}{2x}$ DNE

Case 3 $f(a)$ takes an <u>indeterminate</u> form (like $\frac{0}{0}$)

Ex/ $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} = \lim_{x \to 1} (x^2 + x + 1) = 1^2 + 1 + 1 = 3$

Goal To compute $\lim_{x\to a} f(x)$, use algebra to reduce the limit to Case 1 or 2.

Ex/ $\lim_{x\to 5} \frac{x^3-5x^2}{(x-5)^2} = \lim_{x\to 5} \frac{x^2(x-5)}{(x-5)^2} = \lim_{x\to 5} \frac{x^2}{x-5}$ $\left(\frac{25}{0}\right)$

	$x<5$	5	$x>5$
Top	+		+
Bottom	−		+
Fraction	−		+

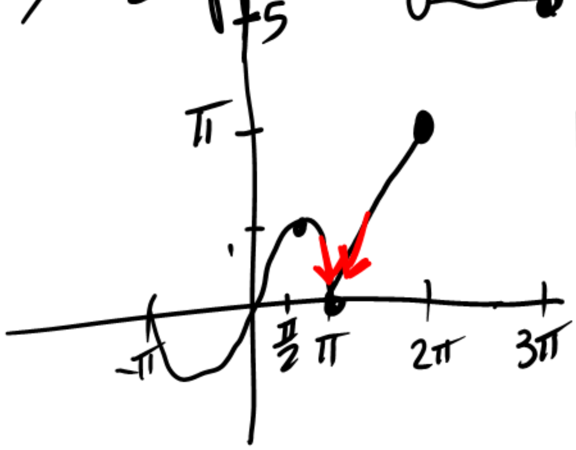
Left-limit is $-\infty$; Right-limit is ∞.

So $\lim_{x\to 5} \frac{x^3-5x^2}{(x-5)^2}$ DNE

Ex/ $\lim_{x\to 2} \frac{x^2-3x+2}{x^2+2x-8} = \lim_{x\to 2} \frac{(x-1)\cancel{(x-2)}}{(x+4)\cancel{(x-2)}} = \lim_{x\to 2} \frac{x-1}{x+4} = \frac{2-1}{2+4} = \frac{1}{6}$

Piecewise Functions

Ex/ Suppose $f(x) = \begin{cases} \sin x & \text{if } -\pi \le x \le \pi \\ x-\pi & \text{if } \pi < x \le 2\pi \\ 5 & \text{if } 2\pi < x \le 3\pi \end{cases}$



To compute $\lim_{x\to a} f(x)$

In this ex., $x = \pi, 2\pi$

Case 1 $x=a$ is not a "boundary point"

Ex/ $\lim_{x\to \frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \sin x = \sin\frac{\pi}{2} = 1$

Case 2 $x=a$ is a "boundary point"

Ex/ $\lim_{x\to\pi} f(x)$:

$\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^-} \sin x = \sin \pi = 0$

$\lim_{x\to\pi^+} f(x) = \lim_{x\to\pi^+} (x-\pi) = \pi - \pi = 0$

So $\lim_{x\to\pi} f(x) = 0$

Ex/ $\lim_{x\to 2\pi} f(x)$:

$\lim_{x\to 2\pi^-} f(x) = \lim_{x\to 2\pi^-} (x-\pi) = 2\pi - \pi = \pi$

$\lim_{x\to 2\pi^+} f(x) = \lim_{x\to 2\pi^+} 5 = 5$

So, $\lim_{x\to 2\pi} f(x)$ DNE

Theorem (Limit Laws) Suppose $a, k, F,$ and G are real numbers. Then if $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$.

- $\lim_{x\to a}\left(k f(x)\right) = k\left(\lim_{x\to a} f(x)\right) = kF$
- $\lim_{x\to a}\left(f(x) \pm g(x)\right) = \left(\lim_{x\to a} f(x)\right) \pm \left(\lim_{x\to a} g(x)\right) = F \pm G$
- $\lim_{x\to a}\left(f(x)g(x)\right) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = FG$
- $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{F}{G}$ (when $G \neq 0$)
- $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n = F^n$ for $n = 1, 2, 3, \ldots$

Ex/ If $\lim_{x\to 3} f(x) = 5$ and $\lim_{x\to 3} g(x) = 2$. Then,

$$\lim_{x\to 3}\left(f(x)+g(x)\right)^2 = \left[\lim_{x\to 3}\left(f(x)+g(x)\right)\right]^2 = \left[\left(\lim_{x\to 3} f(x)\right) + \left(\lim_{x\to 3} g(x)\right)\right]^2$$
$$= [5+2]^2 = 7^2 = 49$$

NOTE: The limit laws only work when the limits of f and g exist.

Ex/ $\lim_{x\to 0} \dfrac{\sin x}{x} = \lim_{x\to 0}\left(\dfrac{1}{x}\right)\sin x = 1$. However, $\lim_{x\to 0} \frac{1}{x}$ DNE. So,

$$\lim_{x\to 0}\left(\frac{1}{x}\right)\sin x \neq \left(\lim_{x\to 0}\frac{1}{x}\right)\left(\lim_{x\to 0}\sin x\right)$$

Ex/ Compute $\lim_{x\to 0} \dfrac{(\cos x - 1)^2 \sin x}{x^2 \cdot x} = \lim_{x\to 0}\left(\dfrac{\cos x - 1}{x}\right)^2 \cdot \dfrac{\sin x}{x}$

$$= \left(\lim_{x\to 0}\frac{\cos x - 1}{x}\right)^2 \cdot \left(\lim_{x\to 0}\frac{\sin x}{x}\right) = 0^2 \cdot 1 = 0$$