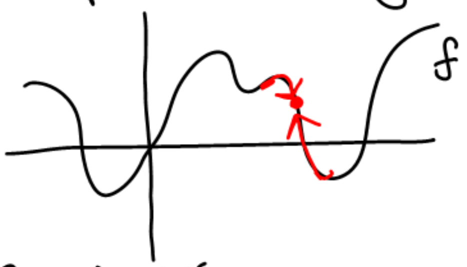
Lecture 4: Continuity

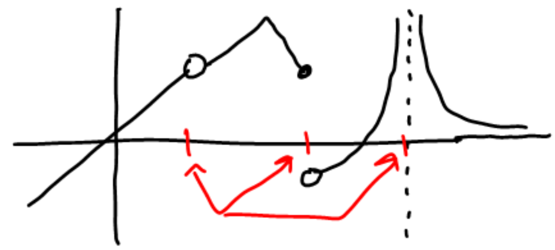
"fcn" or "fn" is shorthand for "function"

GOAL: Be able to determine where and how a function misbehaves.

Intuiatively, a fcn is continuous if there are no abrupt changes in the graph



Continuous



Discontinuous Here

More precisely,

<u>Def</u> A fcn f is <u>continuous</u> at a point c if

$$\lim_{x \to c} f(x) = f(c)$$

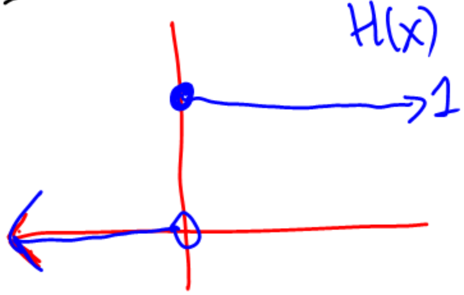
Otherwise, f is discontinuous at c and c is called a discontinuity.

Classifying Discontinuities

There are 3 ways to be continuous

1. $\lim_{x \to c} f(x)$ DNE
2. $f(c)$ is undefined
3. $\lim_{x \to c} f(x) \neq f(c)$

<u>Ex 1</u> Discuss the continuity of $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ at $x=0$.



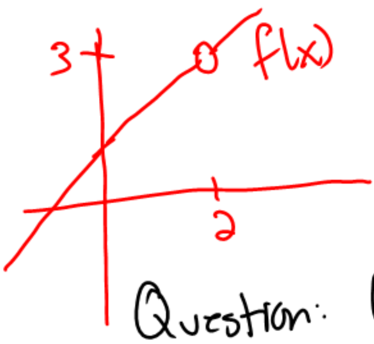
Recall the $\lim_{x \to 0} H(x)$ DNE, but $\lim_{x \to 0^-} H(x) = 0$ and $\lim_{x \to 0^+} H(x) = 1$. So by 1, $H(x)$ is discontinuous at 0.

This is an example of a <u>jump discontinuity</u>.

Ex2 Repeat for $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ at $x = 2$

$f(2) = \frac{0}{0} \Rightarrow f(2)$ is undefined. By ②, f is discontinuous.

$$\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x+1)(x-2)}{x-2} = \lim_{x \to 2} (x+1) = 3$$



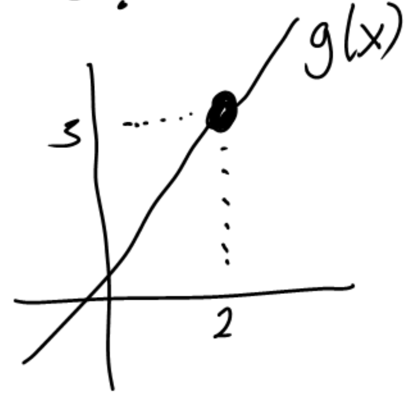
This is an example of a hole (or removable discontinuity) at $x = 2$.

Question: Why is it called removable?

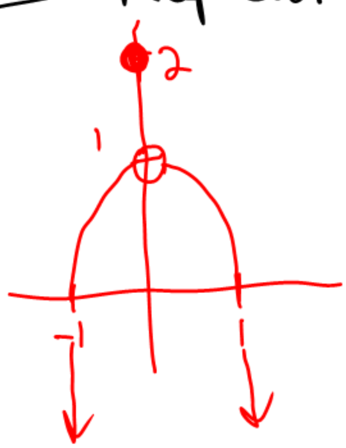
Answer: Let

$$g(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 1} & x \neq 2 \\ 3 & x = 2 \end{cases}$$

↖ Limit Value



Ex3 Repeat for $f(x) = \begin{cases} 1 - x^2 & x \neq 0 \\ 2 & x = 0 \end{cases}$ at $x = 0$



Need to compute

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (1 - x^2) = 1$$

There is a hole at $x = 0$

Ex4 Discuss the continuity of $f(x) = \dfrac{x(x-5)}{x(x-1)}$

Candidates for discontinuities:

$x(x-1) = 0$

$\boxed{x = 0}$ OR $x - 1 = 0$

$\boxed{x = 1}$

Near $x=0$:

$$\lim_{x\to 0} \frac{x(x-5)}{x(x-1)} = \lim_{x\to 0} \frac{x-5}{x-1} = \frac{-5}{-1} = 5$$

$\Rightarrow f$ has a hole at $x=0$

EDITOR NOTE: The left and right limits were flipped in lecture. The current graph is correct.

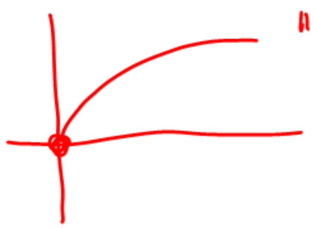
Near $x=1$: $f(1) = \frac{-4}{0} \Rightarrow x$ has a vertical asymptote at $x=1$

The VA is an example of an <u>infinite discontinuity.</u>

<u>Properties of Continuous Functions</u>

(i) We can define "left/right continuity" like what we did with limits.

(ii) It is common to say "a function is continuous on an interval" if it is continuous at every point in the interval.



"$\sqrt{x}$ is continuous on $[0,\infty)$"

<u>Example 5</u> Discuss the continuity of $f(x) = \begin{cases} e^{-x} & x<0 \\ \sqrt{x} & x \geq 0 \end{cases}$

$f(x)$ is continuous on $(-\infty,0) \cup (0,\infty)$

$\cup$ "OR" "Union"

EDITOR NOTE: The graph originally had e^x on the left-side. Current version is correct.

$f(x)$ is right continuous at 0

Jump at $x=0$, don't include it

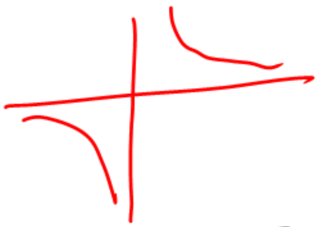
$$\lim_{x\to 0^+} f(x) = 0 = f(0)$$

$f(x)$ is <u>not</u> left continuous: $\lim_{x\to 0^-} f(x) = 1 \neq 0 = f(0)$

Theorem If f and g are continuous at c, then so is

- $f+g$
- $f-g$
- fg
- $\frac{f}{g}$ $(g(c) \neq 0)$
- $k \cdot f$ (k is a constant)

Theorem On their domain, polynomials, rational fcns, root fcns, trig fcns, inverse trig fcns, exp fcns, and log fcns are continuous.



$\frac{1}{x}$ is continuous on its domain

Example 6 [HW4, #11]

$$f(x) = \begin{cases} -6x - \frac{\pi}{2} & \text{if } x \le -\frac{\pi}{2} \\ \cos x & \text{if } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 5\sin x + 9 & \text{if } x \ge \frac{\pi}{2} \end{cases}$$

All fcns are continuous on their part, only could have discontinuities at $x = -\frac{\pi}{2}, \frac{\pi}{2}$

At $-\frac{\pi}{2}$: $\lim_{x \to -\frac{\pi}{2}^-} f(x) = \lim_{x \to -\frac{\pi}{2}^-} \left[-6x - \frac{\pi}{2}\right] = 3\pi - \frac{\pi}{2}$

$\lim_{x \to -\frac{\pi}{2}^+} f(x) = \lim_{x \to -\frac{\pi}{2}^+} \cos x = \cos\left(-\frac{\pi}{2}\right) = 0$

$\Rightarrow$ There is a jump discontinuity at $x = -\frac{\pi}{2}$

At $\frac{\pi}{2}$: $\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \cos x = \cos \frac{\pi}{2} = 0$

$$\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}^+}\left[5\sin x + 9\right] = 5\sin\left(\frac{\pi}{2}\right)+9$$
$$= 5 + 9 = 14$$

$\Rightarrow$ There is a jump discontinuity at $x = \frac{\pi}{2}$

Compositions

Theorem Let f be continuous at G, where $G = \lim_{x\to c} g(x)$. Then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(G)$$

In particular, if g is continuous at c and f is continuous at $g(c)$, then $f(g(x))$ is continuous at c.

Ex 7a

$$\lim_{x\to 1} \frac{x^2(x-1)}{(x^2+3)(x-1)} = \lim_{x\to 1} \frac{x^2}{x^2+3} = \frac{1}{1+3} = \frac{1}{4}$$

7b

$$\lim_{x\to 1} \sqrt{\frac{x^2(x-1)}{(x^2+3)(x-1)}} \underset{\substack{\uparrow \\ \text{cont} \\ \text{at } \frac{1}{4}}}{=} \sqrt{\lim_{x\to 1} \frac{x^2(x-1)}{(x^2+3)(x-1)}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Ex 8

$$\lim_{x\to c} e^{f(x)} = e^{\lim_{x\to c} f(x)}$$