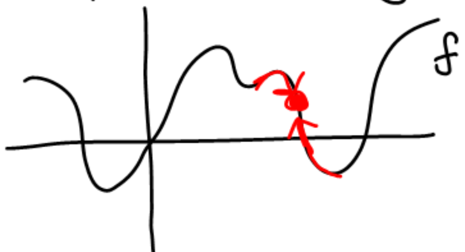
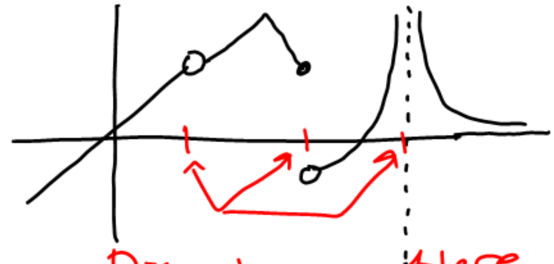
"fcn" or "fn" is shorthand for "function"

GOAL: Be able to determine where and how a function misbehaves.

Intuiatively, a fcn is continuous if there are no abrupt changes in the graph



Continuous



Discontinuous ~~Here~~

More precisely,

<u>Def</u> A fcn f is <u>continuous</u> at a point c if

$$\lim_{x\to c} f(x) = f(c)$$

Otherwise, f is <u>discontinuous</u> at c and c is called a discontinuity.

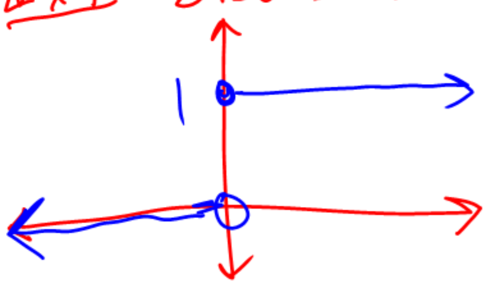
Classifying Discontinuities

There are 3 ways a fcn can be discontinuous

1. $\lim_{x\to c} f(x)$ DNE
2. $f(c)$ is undefined
3. $\lim_{x\to c} f(x) \neq f(c)$

<u>Ex 1</u> Discuss the continuity of $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ at $x=0$

Heaviside



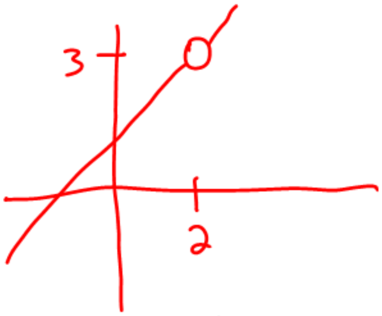
Recall $\lim_{x\to 0} H(x)$ DNE, $\lim_{x\to 0^-} H(x) = 0$ but $\lim_{x\to 0^+} H(x) = 1$. So by 1, $H(x)$ is discontinuous at $x=0$.

This is an example of a <u>jump discontinuity</u>.

Ex 2 Repeat for $f(x) = \dfrac{x^2-x-2}{x-2}$ at $x=2$

$f(2) = \frac{0}{0} \Rightarrow f(2)$ is undefined. f is discontinuous at $x=2$ by (2).

$$\lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2} \frac{(x+1)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x\to 2}(x+1) = 3$$

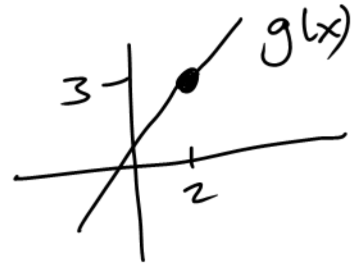


This is an example of a hole or (removable discontinuity) at $x=2$

Why is it called removable? Define a new function

$$g(x) = \begin{cases} \dfrac{x^2-x-2}{x-2} & x \neq 2 \\ 3 & x = 2 \end{cases}$$

↖ Limit Value



Ex 3 Repeat for $f(x) = \begin{cases} 1-x^2 & x \neq 0 \\ 2 & x = 0 \end{cases}$ at $x=0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (1-x^2) = 1$$

$f(0) = 2$

By (3) f has a hole at $x=0$.

Ex 4 Repeat for $f(x) = \dfrac{x(x-5)}{x(x-1)}$

Discontinuities could occur when

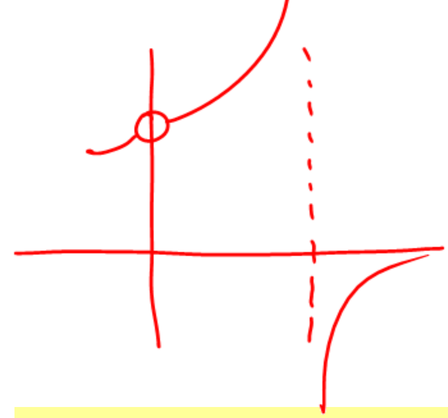
$x(x-1) = 0$

$\boxed{x=0}$ OR $x-1=0 \Rightarrow \boxed{x=1}$

At $x=0$:

$$\lim_{x\to 0} \frac{x(x-5)}{x(x-1)} = \lim_{x\to 0} \frac{x-5}{x-1} = 5$$

$\Rightarrow$ f has a hole at $x=0$



EDITOR NOTE: The left and right limits were flipped in lecture. The current graph is correct.

At $x=1$:

$f(1) = \frac{-4}{0} \Rightarrow f(1)$ is undefined and neither the left nor right limit exists

f has an <u>infinite discontinuity</u> (VA) at $x=1$

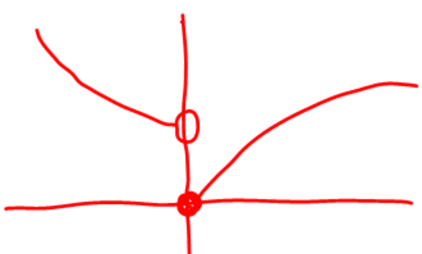
Properties of Continuous Functions

(i) We can define "left/right continuity" like what we did with limits.

(ii) It is common to say "f is continuous on an interval" to mean it is continuous at every point on that interval.

"$\ln x$ is continuous on $(0,\infty)$"

<u>Ex 5</u> Discuss the continuity of $f(x) = \begin{cases} e^{-x} & x<0 \\ \sqrt{x} & x\geq 0 \end{cases}$



$f(0)=0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x} = 0 = f(0)$

$\Rightarrow f$ is right-continuous

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (e^{-x}) = 1 \neq 0 = f(0)$

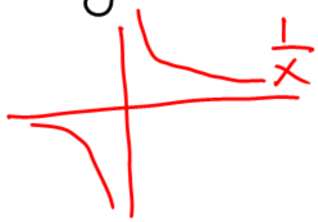
$\Rightarrow$ f is <u>not</u> left continuous

Therefore, f is Continuous on $(-\infty, 0) \cup (0, \infty)$

"union" "OR"

There is a jump at $x = 0$.

<u>Theorem</u> <u>On their domain</u>, polynomials, rational functions, root fcns, trig fcns, inverse trig fcns, exp fcns, and log fcns are continuous.



$\frac{1}{x}$ is Continuous <u>on its domain</u>

<u>Example 6 (HW4, #11)</u>

$$f(x) = \begin{cases} -6x - \frac{\pi}{2} & x \le -\frac{\pi}{2} \\ \cos x & -\frac{\pi}{2} < x \le \frac{\pi}{2} \\ 5\sin x + 9 & x > \frac{\pi}{2} \end{cases}$$

All fcns are continuous on their parts; Candidates for discontinuities are $x = -\frac{\pi}{2}, \frac{\pi}{2}$

At $x = -\frac{\pi}{2}$:

$$\lim_{x \to -\frac{\pi}{2}^-} f(x) = \lim_{x \to -\frac{\pi}{2}^-} \left[-6x - \frac{\pi}{2}\right] = 3\pi - \frac{\pi}{2}$$

$$\lim_{x \to -\frac{\pi}{2}^+} f(x) = \lim_{x \to -\frac{\pi}{2}^+} [\cos x] = \cos\left(-\frac{\pi}{2}\right) = 0$$

$\Rightarrow$ f has a jump discontinuity at $x = -\frac{\pi}{2}$

At $x=\frac{\pi}{2}$: $\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-} [\cos x] = 0$

$$\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}^+} [5\sin x + 9] = 5\sin\tfrac{\pi}{2} + 9$$

$$= 5+9 = 14$$

$\Longrightarrow$ f has a jump discontinuity at $x=\frac{\pi}{2}$

Compositions

Theorem Let f be continuous at $x=G$, where $G=\lim_{x\to c} g(x)$. Then,

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(G)$$

In particular, if g is continuous at c and f is continuous at $g(c)$, then $f(g(x))$ is cont. at c.

Ex 7a

$$\lim_{x\to 1} \frac{x^2(x-1)}{(x^2+3)(x-1)} = \lim_{x\to 1} \frac{x^2}{x^2+3} = \frac{1}{1+3} = \frac{1}{4}$$

Ex 7b

$$\lim_{x\to 1}\sqrt{\frac{x^2(x-1)}{(x^2+3)(x-1)}} \underset{\substack{\uparrow\\ \sqrt{\ } \text{ is}\\ \text{cont. at}\\ \frac{1}{4}}}{=} \sqrt{\lim_{x\to 1}\frac{x^2(x-1)}{(x^2+3)(x-1)}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Ex 8

$$\lim_{x\to c} e^{f(x)} = e^{\lim_{x\to c} f(x)}$$