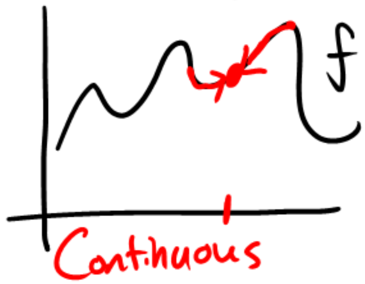
Lecture 4: Continuity

Intuitavely, a function is continuous if there is no abrupt changes in the graph



More precisely,

Def A function f is <u>continuous</u> at a point c if

$$\lim_{x \to c} f(x) = f(c)$$

Otherwise, f is <u>discontinuous</u> at c and the point c is a <u>discontinuity</u>.

Classifying Discontinues

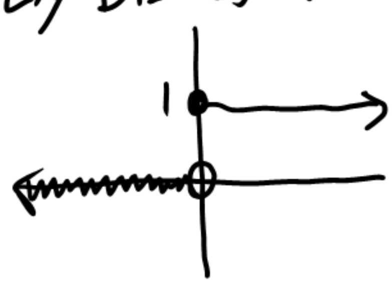
There are 3 ways a function can be discontinuous

① $\lim_{x \to c} f(x)$ DNE

② $f(c)$ undefined

③ $\lim_{x \to c} f(x) \neq f(c)$

EX/ Discuss the continuity of $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ at $x = 0$

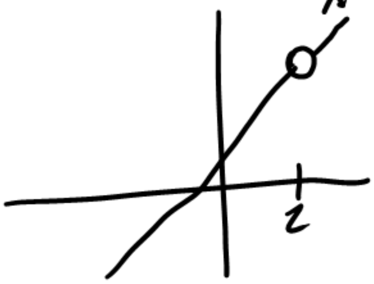


Recall that $\lim_{x \to 0} H(x)$ DNE, so it satisfies ① making H discontinuous at 0.
Continuous everywhere but 0

This is an example of a <u>jump discontinuity</u>.

Ex/ Discuss the continuity of $f(x) = \frac{x^2 - x - 2}{x-2}$ at $x=2$

Sol/ $f(x) = \frac{(x+1)(x-2)}{x-2}$

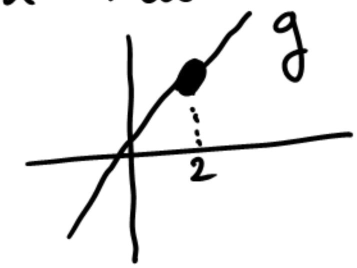


At $x=2$, $f(2)$ is undefined. Hence, satisfy ② making f discontinuous at 2

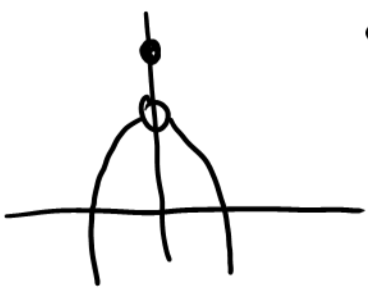
This is an example of a <u>hole</u> (<u>removable discontinuity</u>).

Why is it called removable? Define a new function g

$$g(x) = \begin{cases} f(x) & x \neq 2 \\ \lim_{x \to 2} f(x) = 3 & x = 2 \end{cases}$$



Ex/ Discuss the continuity of $f(x) = \begin{cases} 1-x^2 & x \neq 0 \\ 2 & x = 0 \end{cases}$ at $x=0$



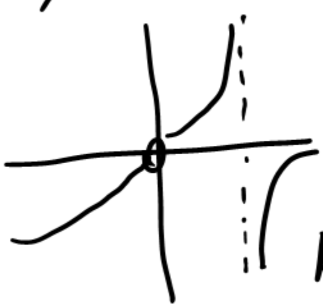
Sol/ $\lim_{x \to 0} f(x) = \lim_{x \to 0} (1-x^2) = 1$

$f(0) = 2$

$\lim_{x \to 0} f(x) \neq f(0)$, so condition ③ makes this a discontinuity.

What type? Hole (removable discontinuity)

Ex/ Discuss the continuity of $f(x) = \frac{x(x-5)}{x(x-1)}$



At $x=0$, there is a hole

At $x=1$, there is a vertical asymptote.

At $x=1$, this is an example of an <u>infinite discontinuity</u>.

Properties of Continuity Functions

Def A function f is left-hand continuous at $x=c$ if

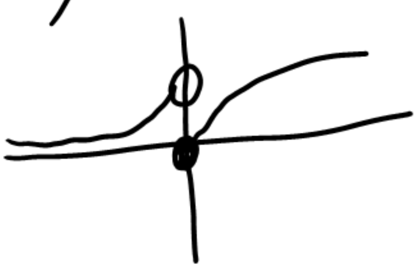
$$\lim_{x\to c^-} f(x) = f(c)$$

Similarly, a function f is right-hand continuous at $x=c$ if

$$\lim_{x\to c^+} f(x) = f(c)$$

Def We say a function is continuous on an interval I if f is continuous for every $c \in I$

Ex/ Discuss the continuity of $f(x) = \begin{cases} e^{-x} & x<0 \\ \sqrt{x} & x \geq 0 \end{cases}$



f is continuous on $(-\infty, 0)$

f is right-hand continuous at $x=0$

f is not left-hand continuous at 0. Why?

$$\lim_{x\to 0^-} f(x) = 1 \neq f(0) = 0$$

f has a jump discontinuity at 0

f is continuous on $(0, \infty)$

Theorem Let f and g be continuous function at $x=c$. Then the following are continuous at $x=c$.

- $f+g$
- $f-g$
- fg
- $\frac{f}{g}$ (when $g(c) \neq 0$)
- af (for some constant a)
- $f \circ g = f(g(x))$

Ex/ When is the function $f(x) = \dfrac{\ln(x) + \arctan(x)}{x^2-1}$ continuous?

Remark: On their domain, polynomials, rational functions, root fcns, trig fcns, inverse trig fcns, exponential fcns, and log fcns are continuous.

$\ln(x)$ is continuous on $(0,\infty)$

$\arctan(x)$ is cont. on $(-\infty,\infty)$

$\ln(x)+\arctan(x)$ is cont. when $x\in(0,\infty)$

x^2-1 is continuous on $(-\infty,\infty)$

$\dfrac{\ln(x)+\arctan(x)}{x^2-1}$ is cont. when ① $x\in(0,\infty)$ ② $x^2-1\neq 0$ $\Rightarrow (0,1)\cup(1,\infty)$

Interchanging Limits

Theorem Let f be a continuous function at $x=c$, and $\lim_{x\to c} g(x)=G$ exists. Then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(G)$$

Ex/ Compute $\lim_{x\to 1}\sqrt{\dfrac{x^2(x-1)}{(x^2+3)(x-1)}}$

Sol/ $$\lim_{x\to 1}\sqrt{\frac{x^2(x-1)}{(x^2+3)(x-1)}} \underset{\substack{\uparrow\\ \sqrt{\ } \text{ cont. at } 1}}{=} \sqrt{\lim_{x\to 1}\frac{x^2(x-1)}{(x^2+3)(x-1)}} = \sqrt{\lim_{x\to 1}\frac{x^2}{x^2+3}}$$

$$= \sqrt{\frac{1}{1+3}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Ex/ $$\lim_{x\to c} e^{f(x)} = e^{[\lim_{x\to c} f(x)]}$$

HW 4 #11

$$f(x) = \begin{cases} -6x - \frac{\pi}{2} & x \le -\frac{\pi}{2} \\ \cos x & -\frac{\pi}{2} < x \le \frac{\pi}{2} \\ 5\sin x + 9 & x \ge \frac{\pi}{2} \end{cases}$$

$$\left.\begin{aligned} &\lim_{x \to -\frac{\pi}{2}^-} f(x) = -6\left(-\frac{\pi}{2}\right) - \frac{\pi}{2} = 3\pi - \frac{\pi}{2} \\ &\lim_{x \to -\frac{\pi}{2}^+} f(x) = \lim_{x \to -\frac{\pi}{2}^+} \cos x = 0 \end{aligned}\right\} \text{Jump discontinuity at } x = -\frac{\pi}{2}$$

For similiar reasons, there is a jump discontinuity at $x = \frac{\pi}{2}$.