Lecture 4: Continuity

Intuitavely, a function is continuous if there is no abrupt changes in the graph

Continuous

Discontinuous at these points

More precisely

Def A function f is continuous at a point c if

$$\lim_{x \to c} f(x) = f(c)$$

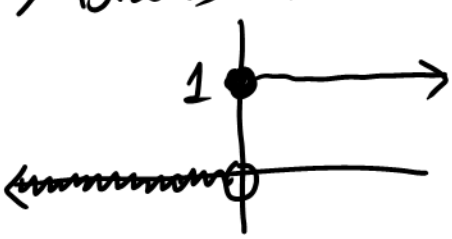
Otherwise, f is discontinuous at c and the point c is called a discontinuity.

Classifying Discontinuites

There are 3 ways a function can be discontinuous

1. $\lim_{x \to c} f(x)$ DNE
2. $f(c)$ is undefined
3. $\lim_{x \to c} f(x) \neq f(c)$

Ex/ Discuss the continuity of $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ at $x = 0$

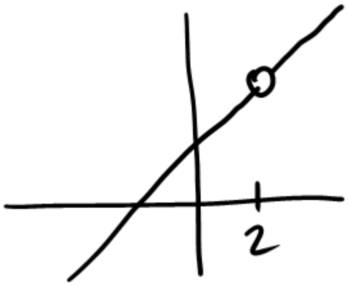


$\lim_{x \to 0} H(x)$ DNE, so H satisfies ① making H discontinuous at 0.

This is an example of a jump continuity.

Ex/ Discuss the continuity of $f(x) = \frac{x^2 - x - 2}{x-2}$ at $x=2$

Sol/ $f(x) = \frac{x^2 - x - 2}{x-2} = \frac{(x+1)(x-2)}{(x-2)}$

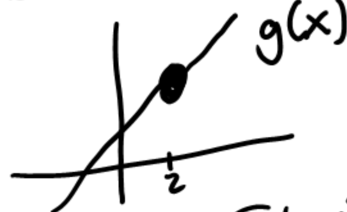


$f(2)$ is not defined, so f satisfies ② making discontinuous at 2.

This is an example of a <u>hole</u> (or <u>removable discontinuity</u>)

<u>Why is it called removable?</u> Define a new function

$$g(x) = \begin{cases} f(x) & x \neq 2 \\ 3 = \lim_{x\to 2} f(x) & x = 2 \end{cases}$$



Ex/ Discuss the continuity of $f(x) = \begin{cases} 1-x^2 & x \neq 0 \\ 2 & x = 0 \end{cases}$ at $x=0$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} (1-x^2) = 1 - 0^2 = 1$

$f(0) = 2$. So $\lim_{x\to 0} f(x) \neq f(0)$

By ③, f has a removable discontinuity (hole) at $x=0$

Ex/ Discuss the continuity of $f(x) = \frac{x(x-5)}{x(x-1)}$ Solve $x(x-1)=0$ to get loc of discontinuites

At $x=0$, $f(0)$ is undefined (takes the form $\frac{0}{0}$)

By ②, f has a hole at $x=0$.

At $x=1$, $f(0)$ is undefined (takes the form $\frac{-4}{0}$)

So by ②, f has a vertical asymptote (or <u>infinite discontinuity</u>) at $x=1$.

Ex/ Find y-coord of $f(x)=\frac{(x^2+3x+2)}{x+1}$ at $x=-1$

Sol/ $f(x)=\frac{(x+2)(x+1)}{(x+1)}$. $f(-1)$ takes the form $\frac{0}{0} \Rightarrow$ hole at $x=-1$

y-coord: It's just $\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{(x+2)\cancel{(x+1)}}{\cancel{(x+1)}} = \lim_{x\to -1}(x+2)$

$= -1+2 = 1$

Properties of Continuous Functions

Def We say f is <u>right-hand continuous</u> at $x=c$ if

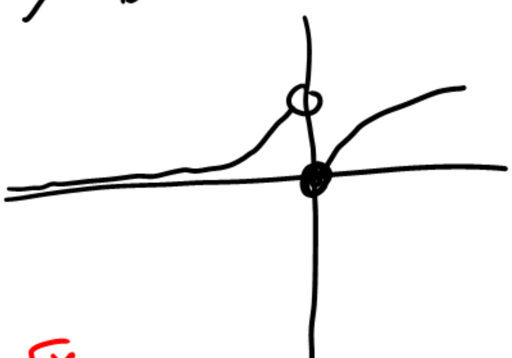
$$\lim_{x\to c^+} f(x) = f(c)$$

Similiarly, f is left-hand continuous at $x=c$ if

$$\lim_{x\to c^-} f(x) = f(c)$$

Def A function is continuous on an interval I if f is continuous at every $c \in I$.

Ex/ Discuss the continuity of $f(x)=\begin{cases} e^{-x} & x<0 \\ \sqrt{x} & x\geq 0 \end{cases}$

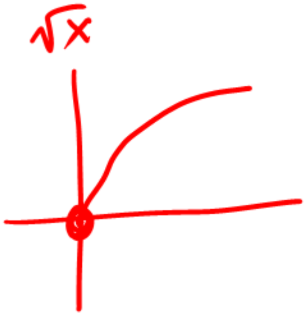


f is continuous on $(-\infty, 0)$

f is right continuous at $x=0$

f is <u>not</u> left continuous

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{-x} = e^0 = 1 \neq 0 = f(0)$$



f has a jump discontinuity at $x=0$
And points

f is continuous on $(0,\infty)$ Continuity (formally) is only valid on open intervals

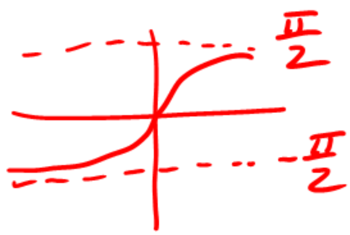
Theorem On their domain, polynomials, rational functions, root fcns, trig fcns, inverse trig fcns, exp. fcns, and log fcns are continuous

Ex  $\frac{1}{x}$ is continuous on its domain $(x \neq 0)$

Theorem If f and g are continuous at $x = c$. Then the following are continuous at c.

- $f+g$
- $f-g$
- fg
- $\frac{f}{g}$ (when $g(c) \neq 0$)
- af (a is constant)
- $f \circ g = f(g(x))$

Ex/ When is $f(x) = \dfrac{\ln(x) + \arctan(x)}{x^2 - 1}$ continuous?



$\ln(x)$ is cont. on $(0, \infty)$

$\arctan(x)$ is cont on $(-\infty, \infty)$

$\ln(x) + \arctan(x)$ is cont on $(0, \infty)$ "x is in the interval $(0,\infty)$"

$x^2 - 1$ is cont. on $(-\infty, \infty)$

$\dfrac{\ln(x) + \arctan(x)}{x^2-1}$ is cont when ① $x \in (0, \infty)$ ② $x^2 - 1 \neq 0 \Rightarrow x \neq 1$ $\Longrightarrow$ $f(x)$ is cont. on $(0,1) \cup (1, \infty)$

Theorem Let f be a continuous function at $x = c$ and $\lim_{x \to c} g(x) = G$ exists. Then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(G)$$

Ex/ Compute $\displaystyle \lim_{x \to 1} \sqrt{\frac{x^2(x-1)}{(x^2+3)(x-1)}} \overset{\text{cont at 1}}{=} \sqrt{\lim_{x \to 1} \frac{x^2(x-1)}{(x^2+3)(x-1)}}$

$$= \sqrt{\lim_{x \to 1} \frac{x^2}{x^2+3}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

① See 9/7 7:30 notes for HW 4 #11

② See webpage for approximating roots PDF